arXiv:1304.2554v1 [cs.PF] 9 Apr 2013

Throughput Optimal Scheduling Policies in Networks of Interacting Queues

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April 10, 2013

Technical Report

Abstract

This report considers a fairly general model of constrained queuing networks that allows us to represent both MMBP (Markov Modulated Bernoulli Processes) arrivals and time-varying service constraints. We derive a set of sufficient conditions for throughput optimality of scheduling policies that encompass and generalize all the previously obtained results in the field. This leads to the definition of new classes of (non diagonal) throughput optimal scheduling policies. We prove the stability of queues by extending the traditional Lyapunov drift criteria methodology.

1 Introduction

Networks of constrained queues have received significant attention from the research community in the last 20 years, since they provide a powerful tool for the analysis of complex systems, such as communication, manufacturing or transportation networks. Specifically, in the context of computer science, networks of constrained queues have been successfully applied to describe packet-level dynamics in wireless networks and in high speed Internet routers whose internal architecture is built around an Input-Queued (IQ) switch.

Throughput and delay are the two basic performance metrics of scheduling policies in a generic constrained queueing or stochastic network. In their pioneering work, Tassiulas and Ephremides [19], have shown that optimal throughput performance can be achieved in networks of constrained queues by employing a

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dynamic scheduling policy according to which, at every time instant, the departure vector maximizes the sum of the "pressures" associated with queues that are served. Where the pressure of a queue q is defined as the difference between its own weight (equal to its own length) and the weight of the queue entered by customers leaving q. The scheme proposed in [19] is referred in the literature as *max scalar, max weight,* or *max pressure* scheduling policy.

Since then, a large body of work has generalized the result in [19], mainly along four lines: i) considering more and more general models of constrained queuing networks; [3, 14] ii) proposing generalizations of the *max scalar* scheduling policy that achieve optimal throughput; [1, 6, 8, 15, 16, 17, 20] iii) looking for simple (low computational) heuristic scheduling policies with throughput guarantees [2, 4, 21]; iv) attempting a characterization of delay properties of throughput optimal scheduling policies [8, 10, 14, 16, 17].

In particular, focusing on the second of the above mentioned aspects, works [1, 5, 6, 8, 11, 15, 16, 17, 20] have shown that the class of throughput optimal scheduling policies is significantly large. It includes low complexity randomized scheduling algorithms [6, 20], as well as, extensions of the *max scalar* scheduling algorithm in which queue weights are possibly non linearly related to queue lengths [1, 8, 11, 16, 17]. Furthermore, in networks of constrained queues with particular symmetry properties, scheduling policies with non diagonal weight assignments (i.e., when the weight of a queue may depend on the length of other queues) have been also shown to be throughput optimal as well [11, 15].

Even if the collection of results already obtained in [1, 6, 8, 11, 15, 16, 17, 20], is rather rich, it is still far from being exhaustive. There are several obscure aspects that prevent full comprehension of the structure of throughput optimal policies. Ideally the long term final objective would be to establish a set of sufficient and *necessary* conditions for throughput optimality of scheduling policies.

This report defines within a unique framework, a set of sufficient conditions for throughput optimality, which encompasses and generalizes all previously known results. Our analysis is based on the application of Lyapunov functions. Our methods, however, substantially differ from prior work because they rely on the application of more general Lyapunov functions, and also involve the adoption of some new stability criteria. For the above reasons we believe that this document provides a valuable contribution toward a deeper understanding of the structure of throughput optimal policies in constrained queuing networks.

This report is organized as follows. In Sect. 2 we introduce system assumptions and notation. Previous work and document contribution are discussed in Sect. 3. Sect. 4 reviews Lyapunov drift criteria that will be invoked in the derivation of our main results. Sect. 5 presents our main findings on throughput optimal scheduling algorithms. At last we conclude the document in Sect. 6.

2 Preliminary definitions and notations

We consider a network composed of N physical queues q_n with $1 \leq n \leq N$, which may represent, for example, either links of a wireless multi-hop network or a virtual output queues (VOQ) in a IQ-switch architecture. The network is traversed by a set $\mathcal{F}$ (with $|\mathcal{F}| = F$) of different customers flows, each-one characterized by a given ingress/egress queue in the network (s_f, d_f).

We assume time to be slotted, and physical queues to have infinite storage capacity. Each physical queue can potentially store customers belonging to several flows. The set of customers belonging to flow f and enqueued in queue q_n forms a virtual queue v_m. The whole network can be regarded as a system of $M \leq FN$ discrete-time virtual queues represented by row vector V, whose m-th element, $1 \leq m < M$ corresponds to virtual queue v_m.

The routes of customer flows in the network are *fixed* (a priori established and time invariant). Without loss of generality, we assume that all customers belonging to flow f and stored in queue v_m will advance to the final destination following the simple path along the network, which corresponds to a predetermined sequence of (virtual/physical) queues to be traversed. We specify network routes by means of an $M \times M$ *routing matrix* $R = [r^{(m,p)}]$ whose element $r^{(m,p)} \in \{0, 1\}$ indicates whether customers departing from virtual queue m enter virtual queue p. [1] We remark that according to our assumptions since all customers of a flow residing in a virtual queue must reach their final destination following the same path queue forking is not permitted. Instead queue joining (i.e., multiple virtual queues feeding into one downstream virtual queue) is permitted.

For any physical queue q_n, function $VQ(n)$ returns the set indices corresponding to the associated virtual queues. For every virtual queue v_m, $PQ(m)$ returns the physical queue that corresponds to v_m. For any virtual queue of index m the function $FL(m)$ returns the index of the corresponding customer flow f. At last for every flow f $FP(f)$ returns the ordered set of indices of virtual queues storing flow f customers along the associated path.

Let $X_t = (x_t^{(1)}, x_t^{(1)}, \ldots, x_t^{(M)})$ be the row vector whose m-th element $x_t^{(m)}$, $1 \leq m \leq M$, represents the number of customers (i.e., either the number of packets or bits/bytes) in queue v_m at time t. The evolution of the number of queued customers is described by $x_{t+1}^{(m)} = x_t^{(m)} + e_t^{(m)} - d_t^{(m)}$, where $e_t^{(m)}$ represents the number of customers that entered virtual v_m in time interval $(t, t+1]$, and $d_t^{(m)}$ represents the number of customers departed from v_m it time interval $(t, t+1]$. $E_t = (e_t^{(1)}, e_t^{(2)}, \ldots, e_t^{(M)})$ is the vector of entrances in the virtual queues, and $D_t = (d_t^{(1)}, d_t^{(2)}, \ldots, d_t^{(M)})$ is the vector of departures from the virtual queues.

[1] In this report the terms server and queue will be used interchangeable.

With this notation, the system evolution equation can be written as:

$$X_{t+1} = X_t + E_t - D_t \tag{1}$$

We represent service constraints among different servers of the network in the following general way. At every time t, the queue departure vector D_t is constrained to lie within a *compact* and *convex* region $\mathcal{D}_t$. We remark that region $\mathcal{D}_t$ may change over time, since it is possibly controlled by a finite state discrete-time Markov chain at steady-state (i.e., $\mathcal{D}_t = \mathcal{D}(S_t^D)$). Without loss of generality we assume $\mathcal{D}(S_t^D)$ to be deterministically associated with the current Markov Chain state S_t^D. We denote by $\mathcal{S}^D$ the state space of Markov Chain S_t^D that models possible variable environmental conditions (such as fading conditions). Additional constraints, such as integrality may be imposed to departure vectors D_t. However, we require that for every state S_t^D, every vertex of $\mathcal{D}(S_t^D)$ represents a feasible departure vector (i.e., a vector that satisfies all constraints). In the particular case in which $\mathcal{D}_t = \mathcal{D}$ (i.e. $\mathcal{D}$ does not vary with time) we say that the system of queues is subject to static service constraints. We observe that this approach is very general and encompasses the classical case [19] in which service constraints are represented by a contention graph. [2] In the latter case $\mathcal{D}$ is defined as convex hull generated by those vectors $D \in \{0,1\}^M$ that correspond to the activation of all possible independent sets of nodes over the contention graph. $D_t \in \{0,1\}^M$, by construction, corresponds to some independent set over the contention graph, and therefore trivially lies in $\mathcal{D}$. Our approach covers also the case in which $\mathcal{D}$ is determined by a rate-power function $\mu(P_t, S_t^D)$ that maps vectors of power allocations to servers P_t (under some constraint on the maximum power that can be employed) into vectors of service rates, for every state S_t^D, as in [14]. In this latter case $\mathcal{D}(S_t^D)$ is to the convex hull generated by service rate vectors corresponding to all possible extremal power allocations.

The entrance vector is the sum of two terms: vector $A_t = (a_t^{(1)}, a_t^{(2)}, \dots, a_t^{(M)})$ representing the customers arrived at the system from outside, and vector $J_t = (j_t^{(1)}, j_t^{(2)}, \dots, j_t^{(M)})$ of recirculating customers; $j_t^{(m)}$ is the number customers departed from some virtual queue and entered virtual queue m in time interval $(t, t+1]$. Note that when customers do not traverse more that one queue (as for a switch in isolation), vector J_t is null for all t, and $A_t = E_t$. In this case we say that the network is traversed by single-hop traffic.

[2] Contentions graphs are typically defined as follows:

Definition 1 *The* contention graph $G_I(\mathcal{V}^I, \mathcal{E}^I)$ *is an undirected graph in which: i) vertices* $v \in \mathcal{V}^I$ *correspond to network (virtual) queues; ii) an edge connects two vertices* v *and* v'*, if the corresponding queues can not simultaneously served.*

Let us consider the external arrival process $A_t = (a_t^{(1)}, a_t^{(2)}, \dots, a_t^{(M)})$; in general we suppose that the sequence A_t is a Markov Modulated Bernoulli Process. We further assume the modulating Markov Chain S_t^A to have a finite number of states. We denote by $\mathcal{S}^A$ its state space. At last we assume the number of arrivals at queues to be deterministically bounded by some constant.[3] We denote by $\Lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(M)})$ the average arrival rate $\mathbb{E}[A_t]$. In the specific case in which A_t forms an i.i.d. sequence, we say that the traffic is i.i.d. The workload W_t provided by customers that in time interval $[t, t+1)$ entered the system of queues is given on average by $W = \mathbb{E}[W_t] = \Lambda(I-R)^{-1}$, being I the identity matrix.

Note that since $J_t = D_t R$, the system evolution equation can thus be rewritten as:

$$X_{t+1} = X_t + A_t - D_t(I-R) \tag{2}$$

At last, given two vectors[4], $A \in \mathbb{R}^M$ and $B \in \mathbb{R}^M$, we denote by $\langle A \cdot B \rangle$ the inner (scalar) product between them $\langle A \cdot B \rangle = AB^T = \sum_{m=1}^{M} a^{(m)} b^{(m)}$, where B^T is the transpose of B; we denote, instead, by $\|A\|$ the Euclidean norm of A, $\|A\| = \sqrt{\langle A \cdot A \rangle}$.

In the following we will use capital letters to denote vectors and matrices, lower case letters to denote scalars, calligraphic characters to denote sets. Moreover we will denote by capital letters, functions of multiple variables while by lower case letters, functions of a single variable; at last, with abuse of notation, given a vector A, we will denote by $f(A)$ the vector whose m-th component is $f(a^{(m)})$.

2.1 Examples

As first example, we consider an input queued switch with N input ports and N output ports. The switching fabric is assumed to be non-blocking and memoryless. Fixed size packets are stored at input ports. Thus one physical queue corresponds to every input port. Each input port maintains a separate virtual queue for each output port. Therefore, the switch in can be modeled as a system comprising $M = N^2$ virtual queues. Let $v_m,\ m = iN + j$ be the virtual queue at input i storing packets directed to output j, with $i, j = 0, 1, 2, \dots, N-1$.

At each time slot, the switch scheduler selects packets to be transferred from input ports to output ports. The set of packets to be transferred during an internal time slot must satisfy two constraints: i) at most one packet can be transferred from each input port, and ii) at most one packet can be transferred toward each output.

[3]We assume that of S_t^D and S_t^A evolve independently, even if this assumption is not strictly needed to obtain our results.

[4]In this report $\mathbb{N}$ denotes the set of non negative integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}^+$ denotes the set of non negative real numbers.

Service constraints can be formalized as are:

$$\sum_{m \in VQ_I(i)} d_t^{(m)} \leq 1 \qquad \sum_{m \in VQ_O(j)} d_t^{(m)} \leq 1 \qquad \forall i, j$$

where $VQ_I(i)$ denotes the set of indices associated to VOQs storing packets at input i; and $VQ_O(j)$, the set of indices of VOQs storing packets directed to output j.

As second example we consider a ad-hoc network with N nodes. Every node is provided with a single transmitter and maintains a per destination virtual queuing structure. Thus, at node i packets destined to node j are enqueued in a virtual queue v_m with $m = iM + j'$. The system of queues can be modeled as a system of $M = N^2$ virtual queues. Packet routes are assumed fixed; all packets at node i destined to node j follow the same route to their destination.

Service constraints come from the fact that; 1) two virtual queues residing in the same node (i.e., insisting on the same physical queue) can not be served simultaneously because of the conflict for the transmitter. 2) some pairs of virtual queues residing in different nodes can not be activated (served) simultaneously because of the mutual interference on the receivers. Service constraints are precisely specified by the associated contention graph $G_I(\mathcal{V}^I, \mathcal{E}^I)$.

2.2 Stability Definitions

Several stability criteria for constrained queuing networks have being defined in the technical literature:

Definition 2 *A system of queues is* rate-stable *if*

$$\lim_{t\to\infty} \frac{X_t}{t} = \lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} (E_i - D_i) = 0. \quad \textit{with probability 1}$$

Definition 3 *A system of queues is* weakly stable *if, for every $\epsilon > 0$, there exists a $b > 0$ such that:*

$$\lim_{n\to\infty} \Pr\{\|X_t\| > b\} < \epsilon$$

where $\Pr\{\mathcal{E}\}$ denotes the probability of event $\mathcal{E}$.

Definition 4 *A system of queues is* strongly stable *if*

$$\limsup_{t\to\infty} \mathbb{E}[\|X_t\|] < \infty$$

Note that strong stability entails weak stability, and that weak stability entails rate-stability. Indeed, rate stability allows queue lengths to indefinitely grow with sub-linear rate, while the weak stability entails that queues are finite with probability 1. This however does not guarantee that the average delay experienced by customers is bounded. Strong stability entails, in addition, the boundedness of average customer delays.

Strong-stability concept can be generalized as follows [5] :

Definition 5 *Given a non-negative continuous function* $F(X) \in C[\mathbb{R}^N \to \mathbb{R}]$, *with* $\lim_{\|X\|\to\infty} F(X) = \infty$*; a system of queues is* $F(X)$-stable *if*

$$\lim_{t\to\infty} \sup \mathbb{E}[F(X_t)] < \infty$$

Note that $F(X)$-*stability* property becomes stricter by selecting functions $F(X)$ that increase faster to ∞, for large $\|X\|$. In other words $F(X)$-*stability* entails $G(X)$-*stability* for any other function $G(X)$ such that [6] $G(X) = O(F(X))$ as [7] $\|X\| \to \infty$. In the following we will make extensive use of the $F(X)$-*stability* criterion.

2.3 Capacity Region

Given a scheduling policy π, the stability region of a network of queues is the set of average arrival vectors Λ in correspondence of which the system is stable (under one of the above criteria). Arrival vector Λ is said to be admissible when it lies in the stability region for some scheduling policy π'. The capacity region of the network is the set of all average admissible arrival vectors, i.e. the set of vectors for which there exists some scheduling policy that makes the system of queues stable. With abuse of language we call admissible an arrival process whose associated rate is admissible.

Under the rate stability criterion, the capacity region of the system $\mathcal{C}_{\rm rate}$, is given by the set of Λ:

[5] $C^k[\mathbb{R} \to \mathbb{R}]$ denotes the class of real valued functions that are k-th times continuously differentiable. Furthermore given a sufficiently smooth function $g(x)$ $\mathbb{R} \to \mathbb{R}$ we denote by $g'(x)$ its first derivative, with $g''(x)$ its second derivative, and with $g^{(h)}(x)$ its h-th derivative.

[6] Given two functions $f(n) \geq 0$ and $g(n) \geq 0$: $f(n) = o(g(n))$ means $\lim_{n\to\infty} f(n)/g(n) = 0$; $f(n) = O(g(n))$ means $\lim\sup_{n\to\infty} f(n)/g(n) = c < \infty$.

[7] For any function $F : \mathbb{R}^{+M} \to \mathbb{R}$ we use $\lim_{\|X\|\to\infty} F(X) = l$ with $l \in \mathbb{R} \cup \{\infty\}$ as shorthand notation to mean that $\lim_{\|\alpha\|\to\infty} F(\alpha X_0) = l$ for any $X_0 \in \mathbb{R}^{+M}$ with $\|X_0\| = 1$

$$\mathcal{C}_{\mathrm{rate}} = \left\{ \Lambda : W = \Lambda(I-R)^{-1} = \sum_{S \in \mathcal{S}^D} \pi_S D(S) \right\} \text{ with } D(S) \in \mathcal{D}(S), \forall S \in \mathcal{S}^D \tag{3}$$

where π_S is the steady state probability associated with states $S \in \mathcal{S}^D$ of the DTMC governing service constraints, and $D(S)$ is an arbitrary vector lying in $\mathcal{D}(S)$ [14, 19]. Observe that $\mathcal{C}_{\mathrm{rate}}$ is a compact (closed and bounded) set in R^{+M}. Under either the weak and strong stability criterion, the capacity region $\mathcal{C}_{\mathrm{weak}} = \mathcal{C}_{\mathrm{strong}}$ corresponds to the interior of $\mathcal{C}_{\mathrm{rate}}$, i.e. to the set of average arrival vectors Λ, whose corresponding workloads W that can be written in the form: $W = \sum_{S \in \mathcal{S}^D} \pi_S D(S)$, with $D(S)$ lying in the interior of $\mathcal{D}(S)$.

3 Previous Work and Document Contribution

In their seminal work, Tassiulas and Ephremides [19] have shown that under i.i.d. arrival processes and static service constraints, optimal throughput can be achieved by employing *max scalar* scheduling policy $\pi_{\max}$, according to which at every time slot t, the departure vector, satisfies:

$$D_t^{\max} = \arg\max_{\substack{D \in \mathcal{D} \\ D \leq X_t}} \langle X_t (I-R)^T \cdot D \rangle$$

More precisely $\pi_{\max}$ guarantees the network of queues to be weakly stable within the capacity region. Observe that the queue length vector X_t has to be interpreted as a vector of *weights* associated to queues, while $X_t(I-R)^T$ is the corresponding vector of *pressures* that takes into account the effect of customers recirculation (for networks of queues supporting single-hop traffic, pressures coincide with weights).

The result in [19] has been extended in several respects. First, the stability properties of the *max scalar* policy have been strengthened (strong stability has been proved) and extended under more general non i.i.d. traffic and dynamic service constraints [3, 14].

Second, the class of throughput optimal schedulers has been extended, including *max scalar* policies that employ non linear queue weights. Under i.i.d. arrival processes and static service constraints, scheduling policies according to which the vector of departures satisfies:

$$D_t^{g} = \arg\max_{\substack{D \in \mathcal{D} \\ D \leq X_t}} \langle g(X)(I-R)^T \cdot D \rangle$$

where $g(x) \in C^1[\mathbb{R}^+ \to \mathbb{R}]$ is a non negative function satisfying: $g(0) = 0$ and $\lim_{t\to\infty} \frac{g'(x)}{g(x)} = 0$, have been shown to be throughput optimal [1, 5, 8, 16, 17, 18]. Particularly relevant are the cases in which $g(X) = X^\alpha$ for $\alpha > 0$. Despite the fact that strong stability has been analytically proved for $\alpha < 1$ very recently [18], it is a longstanding conjecture [8, 16, 17] that optimal delay properties are achieved when $\alpha \to 0$. In [16, 17] this conjecture has been supported by some analytical evidence.

Non-diagonal *max scalar* policies achieving optimal throughput performance, have been have recently identified in [12, 15]. In [15] *Projective Cone Schedulers* PCS, a new class of scheduling policies has been shown to be throughput optimal (under the rate stability criterion) in networks transporting single-hop traffic. According to PCS the departure vector at every time t satisfies:

$$D_t^{PCS} = \arg\max_{\substack{D\in\mathcal{D}\\ D\le X_t}} \langle XQ \cdot D\rangle \tag{4}$$

where Q is a positive definite symmetric matrix with null or negative out of diagonal elements. Observe that according to PCS, contrarily to all previously mentioned schemes, weight associated with queue $v^{(m)}$ may depend on the length of other queues. In this case we say that the scheduling policy employs non diagonal weights. Moreover, we wish to mention that other examples of policies employing non diagonal weights have been earlier shown to achieve throughput optimality in constrained queuing networks with particular structures, such as those corresponding to IQ switches (see for example LPF for IQ switches [7, 11]).

A different, fairly general result has been obtained in [12]. For a general network with static service constraints, given a function $G(X)$, $G \in C^1[\mathbb{R}^{+N} \to \mathbb{R}^+]$, the scheduling policy:

$$D_t^{\nabla G\max} = \arg\max_{\substack{D\in\mathcal{D}\\ D\le X_t}} \langle \nabla G_0(X_t) \cdot D(I - R)\rangle. \tag{5}$$

has been proven to be throughput optimal, provided that $\nabla G(X_t)$ is Lipschitz continuous, $G(X_t)$ is monotonic; i.e. $\nabla G(X_t) \in \mathbb{R}^{+M}$ for any $X_t \in \mathbb{R}^{+M}$ and $||\nabla G(X_t)|| \to \infty$ as $\|X_t\| \to \infty$. Observe, however, that the requirement for $G(X)$ to be monotonic in the previously specified sense, severely reduce the domain of applicability of the result in [12]. For example, non trivial *Projective Cone Scheduler* (with negative out of diagonal elements) are not monotonic. Our analysis generalizes [12] making a further significant step in the direction of the identification of the most general conditions for $G(X)$ that guarantee throughput optimality of the associated *max-scalar* policy.

Scheduling policies with memory [6, 13, 20] represent a further example of throughput optimal schemes for networks with static service constraints. The schemes proposed in [6, 13, 20] are based on the idea of generating an admissible candidate departure vector D_t^c at every slot, according to some simple rule; then the departure vector D_t^{mem} is selected between D_t^c and D_{t-1}^{mem} by maximizing the associated aggregate pressure $D_t^{\mathrm{mem}} = \arg\max\{\langle X \cdot D_t^c \rangle, \langle X \cdot D_{t-1}^{\mathrm{mem}} \rangle\}$. It has been shown that such schemes achieve optimal throughput (i.e., strong stability) under admissible i.i.d. arrival processes and static constraint conditions, provided that at every slot it can be guaranteed $D_t^c = \arg\max \langle X \cdot D \rangle$ with a probability non less than $\delta > 0$. Notice that the above condition is satisfied when D_t^c is uniformly selected among the vertices of $\mathcal{D}$.

This document provides several contributions with respect to previous work: i) Theorem 5 and 6 significantly extend of the class of throughput optimal *max scalar* like policies policies exploiting non linear and non diagonal weights. In particular with respect to [12] , Theorems 5 and 6 do not require $G(X_t)$ to be monotonic. Furthermore throughput optimality is proven under a general model of constrained queuing networks possibly subject to dynamic service constraints and non i.i.d. arrivals. ii) Theorems 7 and 8 generalize the class of throughput optimal scheduling algorithms with memory, applying, for the first time to the best of our knowledge, the concept of schedulers with memory to network of constrained queues subject to dynamic service constraints. iii) Furthermore we are able to strengthen the above results, showing that every polynomial moment of the queue lengths remains finite under any of the above schemes, as long as the average arrival vector lies within the capacity region. iv) At last, from a methodological point of view, we introduce new Foster-Lyapunov drift conditions for $F(X)$-stability (reported in Sect. 4), extending in such a way previous drift arguments.

4 Markov State and Lyapunov Stability Criteria

Under previous assumptions, the process describing the evolution of the system of queues is an irreducible Discrete-Time Markov Chain (DTMC), whose state vector at time t, $Y_t = (X_t, S_t)$, is the combination of vector X_t and a vector S_t that represents the memory of the system in the case in which arrivals are not i.i.d. and/or service constraints are dynamic.

Let $\mathcal{H}$ be the state space of the DTMC, obtained as Cartesian product of the state space [8] $\mathcal{X} \subseteq \mathbb{N}^M$ induced by the queue lengths vector X_t and the state space $\mathcal{S} = \mathcal{S}^A \times \mathcal{S}^D \subset \mathbb{N}^S$ induced by S_t, we further assume $\mathcal{S}$ to be a *finite* state space. Note that $\mathcal{H} \subset \mathbb{N}^{+H}$ with $H = M + S$.

[8] $\mathbb{N}$ denotes the set of non negative integers.

From Definition 3, we can immediately see that DTMC Y_t is positive recurrent, if and only if the system of queues is weakly stable (we recall that the DTMC modelling the system is assumed to be irreducible).

The following general criterion for the (weak) stability of systems is therefore useful in the design of scheduling algorithms. This theorem is a straightforward extension of Foster's Criterion; see [9, 19].

Theorem 1 *Given a system of queues described by a DTMC with state vector $Y_t = (X_t, S_t) \in \mathbb{N}^H$, whose state space $\mathcal{H}$ is the Cartesian product of the denumerable state space $\mathcal{X} \subseteq \mathbb{N}^M$ (with $X_t \in \mathcal{X}$), and a finite state space $\mathcal{S} \in \mathbb{N}^S$ (with $S_t \in \mathcal{S}$); if a lower bounded continuous function $\mathcal{L}(Y)$, called Lyapunov function, $\mathcal{L} : \mathbb{R}^{+H} \to \mathbb{R}$ can be found such that:*

$$\mathbb{E}[\mathcal{L}(Y_{t+1}) \mid Y_t] < \mathcal{L}(Y_t) + v_0 \tag{6}$$

for some $v_0 < \infty$, and

$$\mathbb{E}[\mathcal{L}(Y_{t+1}) - \mathcal{L}(Y_t) \mid Y_t] < -\epsilon \quad \forall Y_t : \|X_t\| > b, \tag{7}$$

for some $\epsilon \in \mathbb{R}^+$ and $b \in \mathbb{R}^+$; then the DTMC is positive recurrent and the system of queues is weakly stable.

Remark: observe that for every $Y_t : \|X_t\| > b$, the satisfaction of (6) immediately follows from (7) (with $v_0 = 0$). Therefore, it is sufficient to verify (6) for $Y_t : \|X_t\| < b$ and (7) to apply the above Theorem. The following result provides a criterion for strong stability.

Theorem 2 *Under the same assumptions of Theorem 1, if $\mathcal{L}(Y)$, additionally satisfies:*

$$\mathbb{E}[\mathcal{L}(Y_{t+1}) - \mathcal{L}(Y_t) \mid Y_t] < -\epsilon\|X_t\| \quad \forall Y_t : \|X_t\| > b, \tag{8}$$

for some $\epsilon \in \mathbb{R}^+$ and $b \in \mathbb{R}^+$; then the system of queues is strongly-*stable.*

Previous stability criteria can be also applied to establish the stability of a DTMC Y_{t_k}, obtained by sampling Y_t in correspondence of an opportunely defined sequence of time instants. In particular we are interested in the case in which $t_k \in \mathbb{N}^+$ form a sequence of stopping times:

Definition 6 *A sequence of random time instants $t_k \in \mathbb{N}^+$ is a sequence of* non-defective *regeneration instants (or stopping times) for the evolution of a system of queues if for any k, the occurrence or non-occurrence of the event $\{t_k = t\}$ depends only on the values of $Y_1, Y_2, Y_3 \cdots Y_t$. Moreover, letting $z_k = t_{k+1} - t_k$, $\mathbb{E}[z_k^h] < \infty$, for any $h \in \mathbb{N}^+$.*

From the strong Markov property [22] immediately follows that the evolution of Markov Chain Y_t after t_k is conditionally independent of the evolution of the system before t_k, given the state $Y(t_k)$, provided that t_k is a stopping time. We remark, instead, that the above conditional independence property does not hold if t_k is a generic random time.

In this case, from the strong stability of Y_{t_k} it is possible to infer strong stability of the original system:

Theorem 3 *Under the same assumptions of Theorem 1, and the additional assumption that both arrival vectors, A_t, and departure vectors, D_t, are bounded in norm, if a lower bounded continuous Lyapunov function $\mathcal{L}(Y)$, $V : \mathbb{R}^{+H} \to \mathbb{R}$ can be found such that for some $v_0 < \infty$, for an opportunely defined* non-defective *sequence of regeneration instants $\{t_k\}$:*

$$\mathbb{E}[\mathcal{L}(Y_{t_{k+1}}) \mid Y_{t_k}] < \mathcal{L}(Y_{t_k}) + v_0 \tag{9}$$

for some $v_0 < \infty$, and

$$\mathbb{E}[\mathcal{L}(Y_{t_{k+1}}) - \mathcal{L}(Y_{t_k}) \mid Y_{t_k}] < -\epsilon\|X_{t_k}\| \quad \forall Y_{t_k} : \|X_{t_k}\| > b \tag{10}$$

for some $\epsilon \in \mathbb{R}^+$ and $b \in \mathbb{R}^+$; then the system of queues is strongly-*stable.*

A brief proof of this statement is in Appendix A.

Lyapunov drift arguments can be extended to obtain the following criterion for $F(X)$-stability:

Theorem 4 *Under the same assumptions of Theorem 1, if a lower bounded continuous Lyapunov function $\mathcal{L}(Y)$, $\mathcal{L} : \mathbb{R}^{+H} \to \mathbb{R}$ can be found satisfying the following two conditions:*

$$\mathbb{E}[\mathcal{L}(Y_{t+1}) \mid Y_t] < \mathcal{L}(Y_t) + v_0 \tag{11}$$

for some $v_0 < \infty$, and

$$\mathbb{E}[\mathcal{L}(Y_{t+1}) - \mathcal{L}(Y_t) \mid Y_t] < -\epsilon F(X_t) \quad \forall Y_t : \|X_t\| > b \tag{12}$$

for some $\epsilon \in \mathbb{R}^+$, $b \in \mathbb{R}^+$, and for a function $F(X) : \mathbb{R}^M \to \mathbb{R}$ non-negative, continuous with $\lim_{\|X\|\to\infty} F(X) = \infty$; then the system of queues is $F(X)$-stable.

The proof is reported in appendix.

At last, using similar arguments as in Theorem 3, we can easily derive the following result:

Corollary 1 *Under the same assumptions of Theorem 1, and the additional assumption that that both arrival vectors A_t and departure vectors D_t are bounded in norm, if a lower bounded continuous Lyapunov function $\mathcal{L}(Y)$, $\mathcal{L} : \mathbb{R}^{+H} \to \mathbb{R}$ can be found such that, for an opportunely defined sequence $\{t_k\}$ of* non-defective *regeneration instants and for some $v_0 < \infty$:*

$$\mathbb{E}[\mathcal{L}(Y_{t_{k+1}}) \mid Y_{t_k}] < \mathcal{L}(Y_t) + v_0, \tag{13}$$

and for some $\epsilon \in \mathbb{R}^+$ and $b \in \mathbb{R}^+$:

$$\mathbb{E}[\mathcal{L}(Y_{t_{k+1}}) - \mathcal{L}(Y_{t_k}) \mid Y_{t_k}] < -\epsilon F(X_{t_k}) \quad \forall Y_{t_k} : \|X_{t_k}\| > b \tag{14}$$

being $F(X) : \mathbb{R}^M \to \mathbb{R}$, a non-negative, continuous function satisfying $\lim_{\|X\|\to\infty} F(X) = \infty$; then the system of queues is $F(X)$-stable.

5 Main Results

In this section we introduce the class of scheduling policies that achieve optimal throughput performance. To improve the readability of the section, all proofs have been moved to Appendix B.

Definition 7 *Given any function $G(X)$, $G \in C^1[\mathbb{R}^{+N} \to \mathbb{R}]$, we define as* $\nabla G(X)$-max scalar*, the scheduling policy $\pi_{\nabla G\max}$ that selects the departure vector according to:*

$$D_t^{\nabla G\max} = \arg\max_{\substack{D\in\mathcal{D}(S_t)\\ D\le X_t}} \langle \nabla G(X_t)(I-R)^T \cdot D\rangle, \tag{15}$$

i.e., the vector of departing customers $D_t^{\nabla G\max} \in \mathcal{D}(S_t)$ is chosen so to maximize the inner product between the departure vector itself, and the gradient of $G(X)$ evaluated at X_t, ($\nabla G(X) \mid_{X=X_t}$, denoted for short by $\nabla G(X_t)$), multiplied by the transpose of matrix $(I - R)$.

Note that $\nabla G(X_t)(I-R)^T$ can be interpreted as the vector of pressures associated with the weight vector $\nabla G(X_t)$. Furthermore, observe that since $\langle \nabla G(X_t)(I - R)^T \cdot D\rangle = \langle \nabla G(X_t) \cdot D(I-R)\rangle$, the $\nabla G(X)$-*max scalar* can be defined as well as scheduling policy according to which:

$$D_t^{\nabla G\max} = \arg\max_{\substack{D\in\mathcal{D}(S_t)\\ D\le X_t}} \langle \nabla G(X_t) \cdot D(I-R)\rangle. \tag{16}$$

At last, in the relevant case in which the network is traversed by single-hop traffic, i.e. when $R = 0$, $D_t^{\nabla G\max}$ satisfies:

$$D_t^{\nabla G\max} = \underset{\substack{D\in\mathcal{D}(S_t)\\ D\leq X_t}}{\arg\max}\langle \nabla G(X_t)\cdot D\rangle. \tag{17}$$

The following two theorems provide conditions for throughput optimality of $\nabla G(X)$-*max scalar* scheduling policies. We denote with $H_G(X)$ the Hessian of $G()$ evaluated at X. We recall that an arrival processes is said admissible if its associated average workload $W = \Lambda(I-R)^{-1}$ lyes in the convex hull of the of the feasible departure vectors, i.e., departure vectors that satisfy service constraints. We denote with $H_G(X)$ the Hessian of G at X

Theorem 5 *The network of queues is $|\nabla G(X)|$-stable under i.i.d. admissible arrival processes and static service constraints, whenever a $\nabla G(X)$-*max scalar *scheduling policy is employed, provided that $G(X)$ is in $C^2[\mathbb{R}^{+N}\to\mathbb{R}]$ and satisfies the following technical conditions:*

1. *$G(X)$ grows to infinity faster than $\|X\|$ when X grows to infinity,* [9] *i.e.:*

$$\lim_{\|X\|\to\infty}\frac{G(X)}{\|X\|} = \infty; \tag{18}$$

2. *$G(X)$ exhibits a sub-exponential behavior for large X; i.e,*

$$\lim_{\|X\|\to\infty}\frac{G(X+Y)}{G(X)} = 1, \quad \lim_{\|X\|\to\infty}\frac{\langle\nabla G(X+Y)\cdot Z\rangle}{\langle\nabla G(X)\cdot Z\rangle} = 1, \quad \lim_{\|X\|\to\infty}\frac{ZH_G(X+Y)Z^T}{ZH_G(X)Z^T} = 1, \tag{19}$$

for arbitrary vectors Y, Z;

3. *the following conditions on the orientation of $\nabla G(X)$ are met:*

$$\langle\nabla G(X)(I-R)^T\cdot D\rangle \leq 0 \quad \forall D\geq 0 \text{ s.t. } \langle X\cdot D\rangle = 0. \tag{20}$$

and

$$\lim_{\|X\|\to\infty}\frac{\langle\nabla G(X)(I-R)^T\cdot D\rangle}{\|\nabla G(X)\|} > 0 \text{ for some } D\geq 0 \tag{21}$$

Stability properties of $\nabla G(X)$-*max scalar* scheduling policies can be extended to more general Markov Modulated Bernoulli Process (MMBP) arrival processes and dynamic service constants, when $G(X)$ satisfies slightly less general conditions:

[9] We recall that for any function $F:\mathbb{R}^{+M}\to\mathbb{R}$ we use $\lim_{\|X\|\to\infty}F(X)=l$ with $l\in\mathbb{R}\cup\{\infty\}$ as shorthand notation to mean that $\lim_{\|\alpha\|\to\infty}F(\alpha X_0)=l$ for any $X_0\in\mathbb{R}^{+M}$ with $\|X_0\|=1$

Theorem 6 *The network of queues is $|\nabla G(X)|$-stable under admissible MMBP arrival processes and general service constraints whenever a $\nabla G(X)$*-max scalar *scheduling policy is employed, provided that $G(X)$ is in $C^\infty[\mathbb{R}^{+N} \to \mathbb{R}]$ and meets the following two conditions:*

1.
$$\limsup_{\|X\|\to\infty} \|(\partial^{h_0} G)(X)\| < \infty \quad \textit{for some } h_0 \in \mathbb{N}; \tag{22}$$

2.
$$\lim_{\|X\|\to\infty} \left\| \frac{(\partial^{h+1} G)(X)}{(\partial^h) G(X)} \right\| = 0 \quad \forall h < h_0; \tag{23}$$

in addition to (18), (20) and (21).

Observe that conditions (22) and (23), which entail (19), express the fact that the dominant behavior of $G(X)$ for $\|X\| \to \infty$ is polynomial.

When $G(X)$ satisfies the technical conditions specified by Theorem 5, we say that it is a weak-potential for the system of queues; we, instead, say that it is a strong-potential for the system of queues, when $G(X)$ satisfies the additional technical conditions specified by Theorem 6. We recall that the proofs of Theorems 5 and 6 are in Appendix B.

Note that according to Theorems 5 and 6, $|\nabla G(X)|$-stability has been proved for $\nabla G(X)$-*max scalar* policies in non overloaded conditions. $|\nabla G(X)|$-stability may become weak, especially when $\nabla G(X)$ increases slowly to infinity for $\|X\| \to \infty$. For example if $G(X) = \frac{1}{1+\alpha} \sum_m (x^{(m)})^{1+\alpha}$ for $\alpha < 1$ (i.e. $\nabla G(X) = X^\alpha$), strong stability of the network of queues is not guaranteed by the above mentioned Theorems. The following Corollary allows to strengthen the assertions of Theorem 5 and Theorem 6, showing that $\nabla G(X)$-*max scalar* policies associated with weak/strong potentials guarantee that every polynomial moment of queue-lengths remains finite within the capacity region:

Corollary 2 *Consider a weak potential function $G(X)$; the network of queues is $\|X\|^h$-stable, for any $h \in \mathbb{N}$ (i.e., every polynomial moment of the queue lengths is finite), under admissible i.i.d. arrival processes and static service constraints, provided that the associated $\nabla G(X)$*-max scalar *scheduling policy is employed. When, instead, $G(X)$ is a strong potential function, $\|X\|^h$-stability can be proved for any $h \in \mathbb{N}$, under MMBP arrival processes and dynamic service constraints.*

Again, we recall that the proof of the Corollary is in Appendix B.

Remark: The class of scheduling policies that satisfy the assumptions of Theorem 5 (or Theorem 6) is fairly large and comprises the following three subclasses of optimal policies, as particular cases. Indeed note that:

1. Any function $G(X)$ in the form: $G(X) = \sum_m = g(x^{(m)})$ where $g(x)$ a function in $C^2[\mathbb{R}^+ \to \mathbb{R}]$ with a super-linear and sub-exponential asymptotic behavior, (i.e. $g(x)$ such that: $\lim_{x\to\infty} \frac{g(x)}{x} = \infty$, $\lim_{x\to\infty} \frac{g(x+1)}{g(x)} = 1$, and $\lim_{x\to\infty} \frac{g'(x)}{g(x)} = \frac{g''(x)}{g'(x)} = 0$), and with the first derivative null in the origin ($g'(0) = 0$) is a weak potential. Furthermore If $g(x)$ is in $C^\infty[\mathbb{R}^+ \to \mathbb{R}]$ and has a polynomial asymptotic behavior for large x, (i.e., $\limsup_{x\to\infty} g^{(h_0)} < \infty$ for some $h_0 \in \mathbb{N}$, and $\lim_{x\to\infty} \frac{g^{(h+1)}(x)}{g^{(h)}(x)} = 0\ \forall h < h_0$), then $G(X)$ is a strong potential. The associated $\nabla G(X)$-*max scalar* policy, according to which $D = \arg\max\langle h(X) \cdot D\rangle$ with $h(x) = g'(x)$ achieves $\|X\|^h$-stability for any h. With abuse of language when $g(X)$ satisfies the above conditions, we say that it is a weak (strong) scalar potential. For this subclass of scheduling policy, we extend findings in [1, 8, 16, 17, 18], since we prove a stronger form of stability (the finiteness of every polynomial moment) under a more general network model with possibly correlated arrivals and dynamic service constraints. As a particular case, if we select $f(x) = \frac{x^{\alpha+1}}{(\alpha+1)}$ we obtain $D_t = \arg\max\langle X^\alpha \cdot D\rangle$. By choosing, instead $f(x) = (x+1)(\log(x+1)-1)$ we can prove stability properties of the scheduling policy according to which $D_t = \arg\max\langle \log(1+X) \cdot D\rangle$.

2. Choosing $G(X) = \langle g(X)Q \cdot g(X)\rangle$ we obtain another subclass of functions satisfying the assumptions of Theorem 6 for networks transporting single-hop traffic, provided that Q is a positive definite symmetric matrix with non positive out-of diagonal elements, and $g(x)$ is $C^\infty[\mathbb{R}^+ \to \mathbb{R}]$, increasing, null in the origin (i.e., $g(0) = 0$) with polynomial asymptotic behavior for large x, (i.e., $\limsup_{x\to\infty} g^{(h_0)} < \infty$ for some $h_0 \in \mathbb{N}$, and $\lim_{x\to\infty} \frac{g^{(h+1)}(x)}{g^{(h)}(x)} = 0\ \forall h < h_0$) and such that $\lim_{x\to\infty} \frac{g(x)}{\sqrt{x}} = \infty$.

   This class of functions extends the class of scheduling policies proposed in [15], for which $D = \arg\max\langle XQ \cdot D\rangle$ (obtained when $g(x) = x$). Once again, we wish to emphasize this class $G(X)$-*max scalar* policies is not covered by [12], since $G(X)$ is not monotonic, as effect of the negative out of diagonal elements of Q.

3. For networks transporting single-hop traffic, every $G(X)$ in the form $G(X) = f(X)PX$ can be easily shown to be a strong potential, provided that: i) P is a positively defined symmetric matrix with strictly positive diagonal elements $p_{mm} > 0$, ii) $f(x)$ is given by:

$$f(x) = \begin{cases} x - 2e^{-\frac{1}{(x-1)^2}} - 2e^{-1} & \text{for } x < 1 \\ x - 2e^{-1} & \text{for } x \geq 1 \end{cases}$$

In particular, the above function satisfies (20) since $f(0) = 0$ and $f'(x) \mid_{x=0} < 0$. This class of policies generalizes the LPF policy [7, 11] defined for input queued switch architectures. To establish a clearer relationship between LPF and the class $\nabla G(X)$-*max scalar* policies with $G(X) = f(X)PX$, we focus on single-hop networks of queues with static service constraints. Without loss of generality, we assume service constraints among virtual queues to be represented by a contention graph. For any virtual queue v_m we can define $\mathcal{I}_m$, the set of virtual queues that are conflicting with v_m. We conventionally assume $v_m \in \mathcal{I}_m$. Then taking matrix P, such that; its element $p_{m,m'} = 1$ if $m' \in \mathcal{I}_m$ (and by construction $m \in \mathcal{I}'_m$) and $p_{m,m'} = 0$ otherwise; we obtain a *max scalar* scheduling policy whose associated queue weights satisfy:

$$w_m = \nabla G(X) \mid_m = \sum_{m:v_m \in \mathcal{I}_m} (x_m - e^{-1}\mathbf{1}_{x_m>0})$$

when v_m is not empty. Weight w_m is instead negative, when v_m is empty.

Now if we consider an IQ switch architecture queue architecture, for any VOQ v_m, $\mathcal{I}_m$ is, by construction, composed of all the virtual queues residing on the same input port or directed to the same output port of the VOQ v_m. Thus, $\nabla G(X)$-*max scalar* scheduling policy associated to $G(X) = f(X)PX$ degenerates into a slightly modified version version of the LPF policy. The difference between the weights of classical LPF and the $\nabla G(X)$-*max scalar* policy for $G(X) = f(X)PX$, is represented by correction terms $e^{-1}\mathbf{1}_{x_m>0}$. At last, observe that $G(X)$ is not monotonic, since weights associated to empty queues are negative.

The above three sub-classes of optimal policies are not at all exhaustive. For example, functions in the form $G(X) = \langle g(X)P \cdot g(X)\rangle$ can be easily proved to be strong potential functions for general constrained networks, provided that P is a symmetric positive defined matrix with strictly positive diagonal elements $p_{mm} > 0$, and $g(x)$ is $C^\infty[\mathbb{R}^+ \to \mathbb{R}]$, increasing, null in the origin (i.e., $g(0) = 0$), with null derivative in the origin (i.e., $g'(0) = 0$), polynomial asymptotic behavior for large x (i.e., $\limsup_{x\to\infty} g^{(h_0)} < \infty$ for some $h_0 \in \mathbb{N}$, and $\lim_{x\to\infty} \frac{g^{(h+1)}(x)}{g^{(h)}(x)} = 0 \ \forall h < h_0$), and such that $\lim_{x\to\infty} \frac{g(x)}{x} = \infty$. Particularly relevant are functions in the form $G(X) = \langle X^{\alpha+1}P \cdot X^{\alpha+1}\rangle$ with $\alpha > 0$.

The following result allows to more precisely characterize the class of well defined potential functions:

Corollary 3 *Given a weak (strong) monotonic potential function $G_1(X)$ (i.e., a*

potential function with $\nabla G_1(X) \geq 0$ for any $X \geq 0$) and a weak (strong) potential function $G_2(X)$, then also

- $G(X) = \alpha G_1(X) + \beta G_2(X) \qquad \forall \alpha, \beta \geq 0$
- $G(X) = G_1(X) G_2(X)$

are weak (strong) potential functions.

Furthermore given $G(X)$, weak (strong) potential function and $g(x) \in C^2[\mathbb{R}^+ \to \mathbb{R}]$, ($g(x) \in C^\infty[\mathbb{R}^+ \to \mathbb{R}]$), increasing with at least linear and sub-exponential (polynomial) asymptotic behavior, i.e. such that: $\liminf_{x\to\infty} \frac{g(x)}{x} > 0$, $\lim_{x\to\infty} \frac{g(x+1)}{g(x)} = 1$, $\lim_{x\to\infty} \frac{g'(x)}{g(x)} = 0$, $\lim_{x\to\infty} \frac{g''(x)}{g'(x)} = 0$ (or $\limsup_{x\to\infty} g^{(h_0)}(x) < \infty$, for some h_0 and $\lim_{x\to\infty} \frac{g^{(h+1)}(x)}{g^{(h)}(x)} = 0$, $\forall h < h_0$,), then $g(G(X))$ is a weak (strong) potential function. If additionally $g'(0) = 0$ also $G(g(X))$ is a weak (strong) potential function.

The proof, which consists in the verification that all conditions of the statement of Theorem 5 (Theorem 6) are met, is rather long and tedious even if conceptually straightforward. For these reasons, we omit it.

Previous corollary characterizes the algebraic structure of potentials and makes the verification of throughput optimality easier for $\nabla G(X)$-*max scalar* policies associated with potentials with complex structure such as: $G(X) = \sum_m g(x^{(m)}) + \langle X^{1+\alpha} P \cdot X^{1+\alpha} \rangle$, $G(X) = \sum_m g(x^{(m)}) \cdot \langle X^{1+\alpha} P \cdot X^{1+\alpha} \rangle$ or $G(X) = g(\langle X^{1+\alpha} P \cdot X^{1+\alpha} \rangle)$, where $g(x)$ is a scalar potential and P is a symmetric positive defined matrix with strictly positive diagonal elements $p_{mm} > 0$.

The following Corollary allows us to further extend the class of throughput optimal scheduling policies:

Corollary 4 *Given a weak (strong) potential function $G(X)$, any scheduling policy $\pi_{\nabla G\mathrm{imp}}$ achieves the same throughput performance (queue stability in non overloaded conditions) of the associated $\pi_{\nabla G\max}$ policy, if it satisfies the following property:*

$$\lim_{\|X\|\to\infty} \langle (D^{\nabla G\max} - D^{\nabla G\mathrm{imp}}) \cdot \nabla G(X)(I-R)^T \rangle = o(\|G(X)\|). \qquad (24)$$

The proof is reported in Appendix B

In general, it is easy to see that scheduling policies according to which:

$$D_t^{\nabla G\max} = \arg\max_{\substack{D\in\mathcal{D}(S_t)\\ D\leq X_t}} \langle \nabla G(Z_t)(I-R)^T \cdot D \rangle \qquad (25)$$

meet constraint (24) as long as $\mathbb{E}[(\|Z_t - X_t\|)]^h$ is bounded for any $h \in \mathbb{N}$. Thus, the class of throughput optimal scheduling policies includes $\nabla G(X)$-*max scalar* policies operating with imperfect/delayed queue status information as well as frame-based $\nabla G(X)$-*max scalar* policies (i.e., policies in which the computation of a new departure vector is not executed at every slot, but just once a while), etc.

5.1 Policies with memory

A further extension to the class of throughput optimal policies can be provided, considering scheduling policies with memory [6, 13, 20]:

Theorem 7 *Given a weak potential function for the system of queues $G(X)$, satisfying:*

$$\lim_{\|X\|\to\infty} \frac{\|H_G(X)\|^\beta}{\|\nabla G(X)\|} = 0 \tag{26}$$

for some $\beta > 1$. The network of queues is $\|X\|^h$-stable for any $h \in \mathbb{N}$ under i.i.d. admissible arrival processes and static service constraints whenever a scheduling policy with memory $\pi_{\nabla G\mathrm{mem}}$ is employed, provided that:

1. *departure vectors selected by $\pi_{\nabla G\mathrm{mem}}$ satisfy the following monotonicity property:*

$$\langle \nabla G(X_{t+1})(I-R)^T \cdot D_{t+1}^{\nabla G\mathrm{mem}} \rangle \;\; \geq \;\; \langle \nabla G(X_{t+1})(I-R)^T \cdot D_t^{\nabla G\mathrm{mem}} \rangle \tag{27}$$

   at every t;

2. *with a probability no smaller then $\delta > 0$, the selected departure vector $D_t^{\nabla G\mathrm{mem}}$ satisfies:*

$$D_t^{\nabla G\mathrm{mem}} = \arg\max_{\substack{D\in\mathcal{D}\\ D\leq X_t}} \langle \nabla G(X_t) \cdot D(I-R) \rangle,$$

   i.e. $D_t = D_t^{\nabla G\max}$ with probability at least δ, at every t.

The proof is reported in Appendix B

As already mentioned it is possible to simply implement a scheduling policy satisfying Theorem 7 requirements by generating at random an admissible candidate departure vector D_t^c, and selecting the departure vector $D_t^{\nabla G\mathrm{mem}}$ according to the rule $D_t^{\nabla G\mathrm{mem}} = \arg\max\{\langle X \cdot D_t^c\rangle, \langle X \cdot D_{t-1}^{\nabla G\mathrm{mem}}\rangle\})$.

Remark: Observe that in this case the space state of the DTMC representing the evolution of the system of queues must be properly defined, by representing also

the information about the last employed departure vector. A natural choice is to take $Y_t = [X_t, D_t^M]$ with $D_t^M = D_t^{\nabla G\text{mem}}$. Further notice that (26) is satisfied whenever $G(X)$ exhibits a polynomial behavior for large $\|X\|$.

When $G(X)$ is a strong potential, previous result can be extended under more general assumptions on arrival processes and service constraints. In this latter case however the complexity of the scheme significantly increases, since the scheduling policy has to memorize the last selected departure vector for every possible state $S \in \mathcal{S}^D$ of the Markov Chain representing service constraints evolution.

Theorem 8 *Let $G(X)$ be a strong potential function of the system of queues. The network of queues is $\|X\|^h$-stable for any $h \in \mathbb{N}$ under admissible MMBP arrival processes and general service constraints, whenever a scheduling policy with memory $\pi_{\nabla G\text{mem}}$ is employed, provided that:*

1. *at every time slot t, the following property is satisfied by departure vectors selected by $\pi_{\nabla G\text{mem}}$:*

$$\langle \nabla G(X_{t+1})(I-R)^T \cdot D_{t+1}^{\nabla G\text{mem}} \rangle \quad \geq \quad \langle \nabla G(X_{t+1})(I-R)^T \cdot D_t^M(S_{t+1}^D) \rangle, \tag{28}$$

   where $D_{t+1}^M(S_{t+1}^D)$ is the departure vector employed by the scheduler $\pi_{\nabla G\text{mem}}$ at the last epoch $t_* < t+1$ in which $S_{t_*}^D = S_{t+1}^D$;

2. *for some $\delta > 0$, the selected departure vector $D_t^{\nabla G\text{mem}}$ satisfies:*

$$D_t^{\nabla G\text{mem}} = \arg\max_{\substack{D \in \mathcal{D}(S_t) \\ D \leq X_t}} \langle \nabla G(X_t) \cdot D(I-R) \rangle,$$

   i.e. $D_t^{\nabla G\text{mem}} = D_t^{\nabla G\text{max}}$ with probability non smaller than δ, at every t.

The proof is reported in Appendix B. Observe that the property expressed by (28) represents the natural extension of (27) to the case dynamic constrains scenario. To satisfy such property, the algorithm has to memorize the last selected departure vector $D_t^{\nabla G\text{mem}}(S)$, for every possible state $S \in \mathcal{S}^D$. Indeed (27) can be obtained by comparing, at time $t+1$ the candidate departure vector D_{t+1}^c and memorized vector $D_t^M(S_{t+1}^D)$.

6 Conclusions

The research on throughput optimal scheduling policies in constrained queuing networks has mainly focused on the analysis of *max scalar* scheduling policies

employing diagonal weights. Only recently [12, 15], the existence of a class of throughput optimal *max scalar* policies employing non diagonal weights has been proved for arbitrary networks. In this document, we have derived a general set of sufficient conditions for throughput optimality that lead to significant extension of results in [12, 15], defining a large body of non diagonal throughput optimal scheduling policies. Furthermore, we have shown, how low complexity scheduling policies with memory can achieve optimal throughput properties under general conditions (i.e., under non i.i.d. arrival processes and dynamic services constraints). This document contributes to make a step toward full comprehension of the structure of throughput optimal scheduling policies in constrained queuing systems. The analysis of delay properties for non diagonal scheduling algorithms is still an important challenging open issue.

References

[1] M. Ajmone Marsan, E. Leonardi, M. Mellia, F. Neri, "On the stability of isolated and interconnected input-queueing switches under multi-class traffic," *IEEE Trans. Inform. Theory*, vol. 51, no. 3, pp. 1167-1174, March 2005.

[2] P. Chaporkar, K. Kar, S. Sarkar, "Throughput Guarantees in Maximal Scheduling in Wireless Networks," *Allerton Conference on Communication, Control and Computing*, 2005.

[3] J. G. Dai, W. Lin, "Maximum Pressure Policies in Stochastic Processing Networks", Operations Research, vol. 53, 2005, pp. 197-218

[4] A. Dimakis, J. Walrand, "Sufficient conditions for stability of longest-queue-first scheduling: Second-order properties using fluid limits," *Adv. Appl. Probab.*, vol. 38, no. 2, pp. 505-521, 2006.

[5] A. Eryilmaz, R. Srikant, J. R. Perkins, "Stable Scheduling Policies for Fading Wireless Channels", *IEEE/ACM Transactions on Netqorking* Vol.13, n 2, pp. 411-424, 2005.

[6] P. Giaccone, B. Prabhakar, D. Shah, "Randomized-scheduling algorithms for high-aggregate bandwidth switches", *IEEE J. Sel. Areas Commun.*, Vol 21, n. 4 pp. 546- 559, May 2003.

[7] G. Gupta, S. Sanghavi, N. Shroff, "Node Weighted Scheduling", *ACM SIGMETRICS, 2009*, June 2009

[8] I. Kesslassy, N. Mckeown, “Analysis of scheduling algorithms that provide 100% throughput in input-queued switches”, *Allerton Conference on Communication Control and Computer,* October 2001.

[9] H.J.Kushner, *Stochastic Stability and Control,* Academic Press, 1967.

[10] E. Leonardi, M. Mellia, F. Neri, M. Ajmone Marsan, “Bounds on delays and queue lengths in input-queued cell switches”, *Journal of ACM*, Vol. 50(4), pp. 520-550, 2003.

[11] A. Mekkittikul, N. McKeown, “A Practical Scheduling Algorithm to Achieve 100% Throughput in Input-Queued Switches”, *INFOCOM*, April 1998.

[12] S. Meyn. (2009), “Stability and asymptotic optimality of generalized maxweight policies,” *SIAM Journal on control and optimization*, no. 47, pp. 3259-3294.

[13] E. Modiano, D. Shah, G. Zussman, “Maximizing Throughput in Wireless networks via Gossiping”, *ACM SIGMETRICS 2006*.

[14] M. J. Neely, E. Modiano, C. E. Rohrs, “Dynamic power allocation and routing for time varying wireless networks,”, *IEEE J. Sel. Areas Commun.*, vol. 23, no. 1, pp. 89-103, Jan. 2005.

[15] K. Ross, N. Bambos. “Projective cone scheduling (PCS) algorithms for packet switches of maximal throughput,” *Transactions on Networking*, Vol. 17, n. 3, pp. 976-989, June 2009.

[16] D. Shah, D. J. Wischik, “Optimal scheduling algorithms for input-queued switches”, *INFOCOM*, April 2006.

[17] D. Shah, D. J. Wischik, “The teleology of scheduling algorithms for switched networks under light load, critical load, and overload”, Technical report, available on line at: \protecthttp://www.cs.ucl.ac.uk/staff/ucacdjw/Research/netsched.html.

[18] D. Shah, J. Tsitsiklis, Y. Zhong, “Qualitative Properties of α-Weighted Scheduling Policies”, *ACM SIGMETRICS 2010*.

[19] L. Tassiulas, A. Ephremides, “Stability properties of constrained queuing systems and scheduling policies for maximum throughput in multi-hop radio networks”, *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1936-1948, Dec. 1992.

[20] L. Tassiulas, “Linear complexity algorithms for maximum throughput in radio networks and input queued switches”, *INFOCOM*, April 1998,

[21] X. Wu, R. Srikant, "Scheduling efficiency of distributed greedy scheduling algorithms in wireless networks", *INFOCOM*, April 2006.

[22] S.P. Meyn and R.L. Tweedie (1993), *Markov chains and stochastic stability*. Cambridge University Press, Second Edition 2009.

A Proofs of Theorems in Section 4

Proof of Theorem 3.

The fact that DTMC Y_{t_k} is strongly stable, i.e., $\limsup_{k\to\infty} \mathbb{E}[\|X_{t_k}\|] < \infty$ is an immediate consequence of (9) and (10) [22]. Then, considering a generic instant t and denoting by $T(t) = \max\{t_k \leq t\}$, we have:

$$\mathbb{E}[\|X_t\|] \leq \mathbb{E}[\|X_{T(t)}\|] + E\left[\| \sum_{\tau=T(t)}^{t-1} \|A_\tau - D_\tau(I-R)\|\right]$$

where $\mathbb{E}[\|\sum_{\tau=T(t)}^{t-1} A_\tau - D_\tau(I-R)\|] \leq \mathbb{E}[\sum_{\tau=T(t)}^{t-1} \|A_\tau - D_\tau(I-R)\|] \leq \mathbb{E}[t-T(t)]c$ where c is an upper bound for $A_t - D_t(I-R)$ (which are bounded by assumption). The assertion follows letting $t\to\infty$. Indeed $\limsup_{t\to\infty} \mathbb{E}[t - T(t)] < \infty$ as consequence of standard renewal arguments. Indeed we recall that t_k is, by assumption, a sequence of non defective regeneration instants (i.e. $\mathbb{E}[z_k^2] = \mathbb{E}[(t_{k+1}-t_k)^2] < \infty$).

Proof of Theorem 4.

Since the assumptions of Theorem 1 are satisfied, every state of the DTMC is positive recurrent and the DTMC is weakly stable. In addition, to prove that the system is $F(X)$-stable, we shall show that $\lim_{t\to\infty} \sup \mathbb{E}[F(X_t)] < \infty$.

Let $\mathcal{H}_b$ be the set of values taken by Y_n for which $\|X_t\| \leq b$ (where (12) does not apply). It is immediate that $\mathcal{H}_b$ is a compact set. Outside this compact set, Equation (12) holds, i.e.

$$\mathbb{E}[\mathcal{L}(Y_{t+1}) - \mathcal{L}(Y_t) \mid Y_t] < -\epsilon F(X_t) \quad \forall Y_t \notin \mathcal{H}_b$$

Considering all Y_t's that do not belong to $\mathcal{H}_b$, we obtain

$$\mathbb{E}[\mathcal{L}(Y_{t+1}) - \mathcal{L}(Y_t) \mid Y_t \notin \mathcal{H}_b] < -\epsilon\, \mathbb{E}[F(X_t) \mid Y_t \notin \mathcal{H}_b]$$

Instead, for $Y_t \in \mathcal{H}_b$, since $\mathcal{H}_b$ is a compact set and $\mathcal{L}(Y)$ continuous we have:

$$\sup_{Y_t \in \mathcal{H}_b} \mathbb{E}[\mathcal{L}(Y_{t+1}) \mid Y_t] \leq \max_{Y_t \in \mathcal{H}_b} \mathcal{L}(Y_t) + v_0 < \infty.$$

Denoting by $c = \max_{Y_t \in \mathcal{H}_b} \mathcal{L}(Y_t) + v_0$ and combining the two previous expressions, we obtain

$$\mathbb{E}[\mathcal{L}(Y_{t+1})] < c\Pr\{Y_t \in \mathcal{H}_b\} + \Pr\{Y_t \notin \mathcal{H}_b\} \cdot \{\mathbb{E}[\mathcal{L}(Y_t) \mid Y_t \notin \mathcal{H}_b] - \epsilon\,\mathbb{E}[F(X_t) \mid Y_t \notin \mathcal{H}_b]\} < \\ < c + \mathbb{E}[\mathcal{L}(Y_t)] - \epsilon\,\mathbb{E}[F(X_t)] + c_0$$

where c_0 is a constant such that $c_0 \geq \{-\mathbb{E}[\mathcal{L}(Y_t) \mid Y_t \in \mathcal{H}_b] + \epsilon\,\mathbb{E}[F(X_t) \mid Y_t \in \mathcal{H}_b]\}\Pr\{Y_t \in \mathcal{H}_b\}$. Note that c_0 can be chosen finite, being $\mathcal{H}_b$ a compact set, and both $F(X)$ and $\mathcal{L}(Y)$ continuous.

By summing over all t from 0 to $\tau_0 - 1$, we obtain

$$\mathbb{E}[\mathcal{L}(Y_{\tau_0})] < \tau_0 c + \mathbb{E}[\mathcal{L}(Y_0)] - \epsilon \sum_{t=0}^{\tau_0-1} \mathbb{E}[F(X_t)] + \tau_0 c_0$$

Thus, for any τ_0, we can write

$$\frac{\epsilon}{\tau_0} \sum_{t=0}^{\tau_0-1} \mathbb{E}[F(X_t)] < c + \frac{1}{\tau_0}\,\mathbb{E}[\mathcal{L}(Y_0)] - \frac{1}{\tau_0}\,\mathbb{E}[\mathcal{L}(Y_{\tau_0})] + c_0$$

$\mathbb{E}[\mathcal{L}(Y_{\tau_0})]$ is lower bounded by definition; assume $\mathbb{E}[\mathcal{L}(Y_{\tau_0})] > c_1$. Hence

$$\frac{\epsilon}{\tau_0} \sum_{t=0}^{\tau_0-1} \mathbb{E}[F(X_t)] < c + \frac{1}{\tau_0}\,\mathbb{E}[\mathcal{L}(Y_0)] - \frac{c_1}{\tau_0} + c_0$$

For $\tau_0 \to \infty$, being $\mathbb{E}[\mathcal{L}(Y_0)]$ and c_1 finite, we can write

$$\limsup_{\tau_0 \to \infty} \frac{\epsilon}{\tau_0} \sum_{t=0}^{\tau_0-1} \mathbb{E}[F(X_t)] < c + c_0$$

The assertion immediately follows.

B Proofs of Theorems in Section 4

Before proceeding with the proofs of the Theorems in Section 5, we recall some standard consequences of Taylor Theorem, of which we will be make extensive use, and we prove three useful Lemmas:

Proposition 1 *Let $G(X)$ $G : [\mathbb{R}^{+N} \to \mathbb{R}]$ be h-times continuously differentiable over an open ball $\mathcal{B}$ centered at a vector X. Then, for any Y such that $X + Y \in \mathcal{B}$,*

$$G(X+Y) = \sum_{i=0}^{h-1} \frac{1}{i} Y^i (\partial^i G)(X) + R_G^{(h)}(X,Y) \tag{29}$$

where the h-order remainder $R_G^{(h)}(X,Y)$ is given by: $R_G^{(h)}(X,Y) = \frac{1}{h!}Y^h(\partial^h G)(X+\beta Y)$, for some $\beta = [0,1]$.

In particular if $G(X)$ is twice continuously differentiable over an open ball $\mathcal{B}$ centered at a vector X, recalling that $\nabla G(X)$ denotes the gradient of G at X, and $H_G(X)$ denotes the Hessian of the function G at X, for any Y such that $X+Y \in \mathcal{B}$, we have:

$$G(X+Y) = G(X) + R_G^{(1)}(X,Y)$$

with $R_G^{(1)}(X,Y) = \langle \nabla G(X+\beta Y) \cdot Y \rangle$ for some $\beta \in [0,1]$, and:

$$G(X+Y) = G(X) + \langle \nabla G(X) \cdot Y \rangle + R_G^{(2)}(X,Y) \tag{30}$$

$R_G^{(2)}(X,Y) = \frac{1}{2}YH_G(X+\beta Y)Y^T$ for some $\beta \in [0,1]$. The above Taylor expansion can be generalized to vectorial functions applying (29) component-wise. In particular we will make use of the following result. Given $G(X)$ twice continuously differentiable over an open ball $\mathcal{B}$ centered at a vector X for any Y such that $X+Y \in \mathcal{B}$, and any $Z \in \mathbb{R}^N$ we have:

$$\langle \nabla G(X+Y) \cdot Z \rangle = \langle \nabla G(X) \cdot Z \rangle + R_{\nabla G}^{(1)}(X,Y,Z) \tag{31}$$

with $R_{\nabla G}^{(1)}(X,Y,Z) = \langle \nabla(\langle \nabla G(X+\beta Y) \cdot Z \rangle) \cdot Y \rangle = \frac{1}{2}ZH_G(X+\beta Y)Y^T$ for some $\beta \in [0,1]$.

Lemma 1 *If $G(X)$ satisfies the conditions of Theorem 5, then:*

$$\lim_{\|X\| \to \infty} \langle \nabla G(X) \cdot \hat{X} \rangle = \infty$$

$\hat{X}$ being the normalized vector parallel to X

Proof: The proof can be immediately obtained by applying l'Hopital's rule to the indefinite form (18):

$$\lim_{\alpha \to \infty} \frac{G(X)}{\|X\|} = \lim_{\alpha \to \infty} \frac{G(\alpha \hat{X})}{\alpha}$$

and recalling that $\lim_{\alpha \to \infty} \langle \nabla G(\alpha \hat{X}) \cdot \hat{X} \rangle = \lim_{\|X\| \to \infty} \langle \nabla G(X) \cdot \hat{X} \rangle$ exists in light of (19). Observe as immediate consequence of previous statement we get:

$$\lim_{\|X\| \to \infty} \|\nabla G(X)\| = \infty$$

Lemma 2 *If $G(X)$ satisfies the conditions of Theorem 5 then:*

$$G(X+Y)-G(X)=R^{(1)}(X,Y)=\begin{cases} O(||\nabla G(X)||) \\ o(G(X)) \end{cases} \qquad \text{as } \|X\|\to\infty, \tag{32}$$

whenever Y is an arbitrary vector. If $G(X)$ satisfies the conditions of Theorem 6.

$$\mathbb{E}[G(X+Y)]-G(X)=\mathbb{E}[R^{(1)}_G(X,Y)]=\begin{cases} O(||\nabla G(X)||) \\ o(G(X)) \end{cases} \qquad \text{as } \|X\|\to\infty, \tag{33}$$

whenever Y is a random vector with finite polynomial moments $\mathbb{E}[||Y||^h]<\infty\,\forall h$.

Similarly, if $G(X)$ satisfies the conditions of Theorem 5

$$\langle\nabla G(X+Y),Z\rangle-\langle\nabla G(X),Z\rangle=R^{(1)}_{\nabla G}(X,Y,Z)=\begin{cases} O(||H_G(X)||) \\ o(\|\nabla G(X)\|) \end{cases} \qquad \text{as } \|X\|\to\infty, \tag{34}$$

whenever Z and Y are two arbitrary vectors. If $G(X)$ satisfies the conditions of Theorem 6

$$\mathbb{E}\langle\nabla G(X+Y),Z\rangle-\langle\nabla G(X),Z\rangle=\mathbb{E}[R^{(1)}_{\nabla G}(X,Y,Z)]=\begin{cases} O(||H_G(X)||) \\ o(\|\nabla G(X)\|) \end{cases} \qquad \text{as } \|X\|\to\infty, \tag{35}$$

whenever Z is an arbitrary vector and Y is a random vector with finite polynomial moments $\mathbb{E}[||Y||^h]<\infty$) $\forall h$.

At last, if $G(X)$ satisfies the conditions of Theorem 5:

$$R^{(2)}(X,Y)=O(\|H_G(X)\|) \qquad R^{(2)}(X,Y)=o(\|\nabla G(X)\|) \tag{36}$$

for any vector Y. If $G(X)$ satisfies the conditions of Theorem 6:

$$\mathbb{E}[R^{(2)}(X,Y)]=O(\|H_G(X)\|) \qquad \mathbb{E}[R^{(2)}(X,Y)]=o(\|\nabla G(X)\|) \tag{37}$$

whenever Y is a random vector with finite polynomial moments $\mathbb{E}[||Y||^h]<\infty\,\forall h$.

Proof:

Properties (32) and (34) are an immediate consequence of the sub-exponential behavior of $G(X)$, i.e (19). Now focusing on (33), observe that expanding $G(X)$ in Taylor series around X, we obtain:

$$\mathbb{E}[G(X+Y)]=G(X)+\mathbb{E}[\sum_{i=1}^{h_0-1}\frac{1}{i}Y^i(\partial^i G)(X)]+\mathbb{E}[R^{(h_0)}_G(X,Y)]$$

where $\mathbb{E}[R_G^{(h_0)}(X,Y) = \frac{1}{h_0!}\mathbb{E}[Y_0^h(\partial^h G)(X+\beta Y)] \leq \frac{1}{h_0!}\mathbb{E}[\|Y_0^h\|]\sup_{Z\in\mathbb{R}^{+N}}\|(\partial^h G)(Z)\| < \infty$, because by assumptions $\mathbb{E}[Y_0^h]$ is bounded as well as $\sup_{Z\in\mathbb{R}^{+N}}\|(\partial^h G)(Z)\| < \infty$ (recalling (22)). Thus the last term is negligible with respect to $G(X)$ and $\|\nabla G(X)\|$ since both $G(X)\to\infty$ (by hypothesis) and $\|\nabla G(X)\|\to\infty$ (by Lemma 1) as $\|X\|\to\infty$ b). Furthermore, $\mathbb{E}[\sum_{i=1}^{h_0-1}\frac{1}{i}Y^i(\partial^i G)(X)] = \sum_{i=1}^{h_0-1}\frac{1}{i}\mathbb{E}[Y^i](\partial^i G)(X) = O(\|\nabla G(X)\|) = o(G(X))$, since $\mathbb{E}[Y^i]<\infty$ and $\|(\partial^i G)(X)\| = o(\partial^{i-1}G(X))$ for any $1\leq i<h_0$, from (23). Thus (33) is proved. (35) can be proved repeating the same arguments to every component of $\nabla G(X)$.

(36) can be proved observing that by definition $R_{\nabla G}^{(1)}(X,Y,Y)$ and $R_G^{(2)}(X,Y)$ are closely related, indeed: $R_{\nabla G}^{(1)}(X,Y,Y) = YH_G(X+\beta_1 Y)Y^T$ for a $\beta_1\in[0.1]$, while $R_{\nabla G}^{(2)}(X,Y) = YH_G(X+\beta_2 Y)Y^T$ for a $\beta_2\in[0.1]$, possibly different from β_1. Now by (19) we get that $\lim_{\|X\|\to\infty}\frac{R_G^{(2)}(X,Y)}{R_{\nabla G}^{(1)}(X,Y,Y)} = \lim_{\|X\|\to\infty}\frac{YH_G(X+\beta_2 Y)Y^T}{YH_G(X+\beta_1 Y)Y^T} = 1$, furthermore from (34) we have $\lim_{\|X\|\to\infty}\frac{R_{\nabla G}^{(1)}(X,Y,Y)}{\|\nabla G(X)\|} = 0$, (or in alternative $\liminf_{\|X\|\to\infty}\frac{R_{\nabla G}^{(1)}(X,Y,Y)}{\|H_G(X)\|} > 0$ and $\limsup_{\|X\|\to\infty}\frac{R_{\nabla G}^{(1)}(X,Y,Y)}{\|H_G(X)\|} > \infty$) thus combining both we get: $\lim_{\|X\|\to\infty}\frac{R_G^{(2)}(X,Y)}{\|\nabla G(X)\|} = 0$ (or in alternative $\liminf_{\|X\|\to\infty}\frac{R_{\nabla G}^{(2)}(X,Y)}{\|H_G(X)\|} > 0$ and $\limsup_{\|X\|\to\infty}\frac{R_G^{(2)}(X,Y)}{\|H_G(X)\|} > \infty$)

At last (37) can be proved observing that: $\mathbb{E}[G(X+Y)] = G(X)+\langle\nabla G(X)\cdot\mathbb{E}[Y]\rangle+\mathbb{E}[R_G^{(2)}(X,Y)] = G(X)+\langle\nabla G(X)\cdot\mathbb{E}[Y]\rangle+\mathbb{E}[\sum_{i=2}^{h_0-1}\frac{1}{i}Y^i(\partial^i G)(X)]+\mathbb{E}[R_G^{(h_0)}(X,Y)]$ thus:

$$\mathbb{E}[R_G^{(2)}(X,Y)] = \mathbb{E}[\sum_{i=2}^{h_0-1}\frac{1}{i}Y^i(\partial^i G)(X)] + \mathbb{E}[R_G^{(h_0)}(X,Y)]$$

Now from (22) and (23), as before, we can conclude that all terms on the right are $O(\|H_G G(X)\|)$ and $o(\|\nabla G(X)\|)$.

Lemma 3 *If $G(X)$ satisfies the conditions of Theorem 5 (and in particular condition (20)), then:*

$$\max_{\substack{D\in\mathcal{D}\\ D\leq X_t}}\langle\nabla G(X_t)(I-R)^T\cdot D\rangle \geq \max_{D\in\mathcal{D}}\langle\nabla G(X_t)(I-R)^T\cdot D\rangle + o(\nabla G(X_t)) \quad (38)$$

i.e, there is always an "almost" optimal feasible departure vector satisfying the condition $D_t\leq X_t$ among the departure vectors that maximize the scalar product $\langle\nabla G(X_t)(I-R)^T\cdot D\rangle$.

Proof: We denote by $\tilde{D}_t^{\nabla G\max} = \arg\max_{D\in\mathcal{D}}\langle \nabla G(X_t)(I-R)^T \cdot D\rangle$, and by $D_t^{(1)} = \min(\tilde{D}_t^{\nabla G\max}, X_t)$. Note that by construction:

$$\langle \nabla G(X_t)(I-R)^T \cdot D_t^{(1)}\rangle \;\leq\; \langle \nabla G(X_t)(I-R)^T \cdot D_t^{\nabla G\max}\rangle \tag{39}$$

Furthermore note that by construction $X_t^{(1)} = X_t - D_t^{(1)}$ and $\tilde{D}_t^{\nabla G\max} - D_t^{(1)}$ are orthogonal since the non null components of $\tilde{D}_t^{\nabla G\max} - D_t^{(1)} = \max(\tilde{D}_t^{\nabla G\max} - X_t, 0)$ coincide with the null of $X_t^{(1)} = \max(X_t - \tilde{D}_t^{\nabla G\max}, 0)$. Thus according to (20):

$$\langle \nabla G(X_t^{(1)})\cdot(\tilde{D}_t^{\nabla G\max} - D_t^{(1)})(I-R)\rangle = \langle \nabla G(X_t^{(1)})(I-R)^T \cdot \tilde{D}_t^{\nabla G\max} - D_t^{(1)}\rangle \leq 0 \tag{40}$$

now expanding in Taylor series $\nabla G(X)$ around point X_t we obtain

$$\nabla G(X_t^{(1)}) = \nabla G(X_t) + R^{(1)}_{\nabla G}(X_t, -D_t^{(1)})$$

Since $D_t^{(1)}$ is norm bounded, from (34) we can conclude that the remainder $R^{(1)}_{\nabla G}(X_t, -D_t^{(1)})$ is $o(\nabla G(X_t))$ and thus:

$$\langle \nabla G(X_t)\cdot(\tilde{D}_t^{\nabla G\max} - D_t^{(1)})(I-R)\rangle = \langle \nabla G(X_t^{(1)})\cdot(\tilde{D}_t^{\nabla G\max} - D_t^{(1)})(I-R)\rangle + o(\nabla G(X_t)) \tag{41}$$

from which the assertion follows recalling (39) and (40). Indeed

$$\langle \nabla G(X_t)\cdot(\tilde{D}_t^{\nabla G\max} - D_t^{\nabla G\max})(I-R)\rangle \overset{(39)}{\leq} \langle \nabla G(X_t)\cdot(\tilde{D}_t^{\nabla G\max} - D_t^{(1)})(I-R)\rangle$$
$$\overset{(41)}{=} \langle \nabla G(X_t^{(1)})\cdot(\tilde{D}_t^{\nabla G\max} - D_t^{(1)})(I-R)\rangle + o(\nabla G(X_t)) \overset{(40)}{\leq} o(\nabla G(X_t))$$

Proof of Theorem 5.

First, observe that since arrivals are assumed i.i.d. and service constraints are assumed to be static, we have $\mathcal{H} = \mathcal{X}$.

The idea of the proof is rather simple; $G(X)$ can be interpreted as a Lyapunov function for the system. The stability of the network of queues follows from the fact that the drift conditions of Theorem 4 can be verified.

We now evaluate the drift of $G(X_t)$ for large values of X_t. By definition:

$$\Delta\mathcal{L} = \mathbb{E}[G(X_{t+1}) - G(X_t) \mid X_t] = = E\left[G\left(X_t + A_t - D_t^{\nabla G\max}(I-R)\right)\middle| X_t\right] - G(X_t)$$

and approximating $G(X_t + A_t - D_t^{\nabla G\max}(I-R))$ with its Taylor polynomial centered at X_t, we get:

$$
\begin{aligned}
& E\left[G\left(X_t + A_t - D_t^{\nabla G\max}(I-R)\right) \big| X_t\right] \\
&= G(X_t) + \langle \nabla G(X_t) \cdot E\left[\left(A_t - D_t^{\nabla G\max}(I-R)\right)\right]\rangle + R_G^{(2)}\left(X_t, A_t - D_t^{\nabla G\max}(I-R)\right)
\end{aligned} \tag{42}
$$

where the remainder $R_G^{(2)}(X_t, A_t - D_t^{\nabla G\max}(I-R))$, since both A_t and $D_t^{\nabla G\max}$ are a bounded norm vectors, in light of (36) satisfies:

$$
\lim_{\|X_t\| \to \infty} \frac{\left|\left|R_G^{(2)}\left(X_t, A_t - D_t^{\nabla G\max}(I-R)\right)\right|\right|}{\|\nabla G(X_t)\|} = 0
$$

thus:

$$
\begin{aligned}
& E\left[G\left(X_t + A_t - D_t^{\nabla G\max}(I-R)\right) \big| X_t\right] \\
& \quad = G(X_t) + \langle \nabla G(X_t) \cdot E\left[\left(A_t - D_t^{\nabla G\max}(I-R)\right)\right]\rangle + o(|\nabla G(X_t)|)
\end{aligned} \tag{43}
$$

Where, $\langle \nabla G(X_t) \cdot \mathbb{E}[(A_t - D_t^{\nabla G\max}(I-R))]\rangle = \langle \nabla G(X_t) \cdot \Lambda - D_t^{\nabla G\max}(I-R)\rangle = \langle \nabla G(X_t) \cdot \Lambda\rangle - \langle \nabla G(X_t) \cdot D_t^{\nabla G\max}(I-R)\rangle$. Since by assumption $\Lambda(I-R)^{-1}$ lies in the interior of $\mathcal{D}$, an $\epsilon' > 0$ can be found, such that also $\Lambda(I-R)^{-1} + \epsilon' D_t^{\nabla G\max}$ lies in $\mathcal{D}$, we obtain:

$$
\begin{aligned}
& \langle \nabla G(X_t) \cdot D_t^{\nabla G\max}(I-R)\rangle = \max_{\substack{D \in \mathcal{D} \\ D \leq X_t}} \langle \nabla G(X_t)(I-R)^T \cdot D\rangle = \max_{D \in \mathcal{D}} \langle \nabla G(X_n)(I-R)^T \cdot D\rangle + o(\|\nabla G(X_t)\|) \\
\geq \ & \langle \nabla G(X_t)(I-R)^T \cdot \Lambda(I-R)^{-1} + \epsilon' D_t^{\nabla G\max}\rangle \geq \langle \nabla G(X_t) \cdot \Lambda\rangle + \epsilon \|\nabla G(X_t)\|
\end{aligned} \tag{44}
$$

for a suitable $\epsilon > 0$. Observe that the second equation holds by virtue of Lemma 3, while the last holds since $\frac{\langle \nabla G(X_t)(I-R)^T \cdot D_t^{\nabla G\max}\rangle}{\|\nabla G(X_t)\|} = \frac{\epsilon}{\epsilon'} > 0$ by virtue of (21).

In conclusion:

$$
E\left[G(X_{t+1}) - G(X_t) \mid X_t\right] \leq -\epsilon \|\nabla G(X_t)\| + o(\|\nabla G(X_t)\|)
$$

for large X_t, and therefore (12) is satisfied since for any $\epsilon'' < \epsilon'$, a $b > 0$ can be found such that:

$$
E\left[G(X_{t+1}) - G(X_t) \mid X_t\right] \leq -\epsilon'' \|\nabla G(X_t)\|
$$

for $\|X_t\| > b$.

Furthermore for any $X_t : \|X_t\| \leq b$, $G(X_{t+1}) - G(X_t) = G(X_t + A_t - D_t^{\nabla G \max}(I-R)) - G(X_t)$ is bounded. Indeed once again we recall vector $\|A_t - D_t^{\nabla G \max}(I-R)\|$ is bounded in norm. Let ζ be a bound for $\|A_t - D_t^{\nabla G \max}(I-R)\|$ Now $\|X_{t+1}\|\| = \|X_t + A_t - D_t^{\nabla G \max}(I-R)\| \leq \|X_t\| + \|A_t - D_t^{\nabla G \max}(I-R)\| \leq b + \zeta$

Thus being $G(X)$ continuous, and thus bounded over compact domains both from above and below: $G(X_{t+1}) - G(X_t) \leq \max_{X_t:\|X_t\|\leq b+\zeta} G(X) - \min_{X_t:\|X_t\|\leq b} G(X)$. The $|\nabla G(X)|$-stability of the system of queues immediately follows since $\lim_{\|X\|\to\infty} \nabla G(X) = \infty$ (as result of Lemma 1)

Proof of Theorem 6.

The generalization to the case in which S_t is a non trivial Markov Chain can be carried out by sampling the process Y_t in correspondence of the instants $\{t_k\}$ at which $S_{t_k} = S_0$ for some specific state S_0. From theory of DTMC (recalling that S_t has a finite number of states) immediately follows that $\{t_k\}$ forms a sequence of non defective regeneration points for the system. Thus applying Corollary 1 we can prove the stability of the system of queues. To simplify the notation we assume traffic to be single-hop along our proof; however we wish to emphasize that the proof for the more general case goes exactly along the same lines and can easily recovered by replacing in the following derivation the departure vector at time t, $D_t^{\nabla G \max}$ with $D_t^{\nabla G \max}(I-R)$.

Again we select $G(X)$ as Lyapunov function. Approximating $G(X)$ with its second order Taylor s expansion, we get:

$$
\begin{aligned}
&\mathbb{E}[G(X_{t_{k+1}}) \mid Y_{t_k}] \\
&= G(X_{t_k}) + < \nabla G(X_{t_k}) \cdot \sum_{t_k}^{t_{k+1}-1} (\mathbb{E}[A_t - D_t^{\nabla G \max})] > + \mathbb{E}\left[R_G^2\left(X_{t_k}, \sum_{t_k}^{t_{k+1}-1} (A_t - D_t^{\nabla G \max}\right)\right]
\end{aligned}
\tag{45}
$$

Now, since all polynomial moments of vector $\sum_{t_k}^{t_{k+1}-1}(A_t - D_t^{\nabla G \max})$ are, by construction, finite, (this because every vector $A_t - D_t^{\nabla G \max}$ is bounded in norm and polynomial moments of $z_k = t_{k+1} - t_k$ are finite,) from (37) we obtain that $\mathbb{E}\left[R_G^2\left(X_{t_k}, \sum_{t_k}^{t_{k+1}-1}(A_t - D_t^{\nabla G \max}\right)\right] = o(\|\nabla G(X_{t_k})\|)$, i.e.,

$$\mathbb{E}[G(X_{t_{k+1}}) \mid Y_{t_k}] = G(X_{t_k}) + \langle \nabla G(X_{t_k}) \cdot \mathbb{E}\left[\sum_{t_k}^{t_{k+1}-1} (A_t - D_t^{\nabla G\max})\right] \rangle + o(\|\nabla G(X_{t_k})\|) \tag{46}$$

Furthermore:

$$\langle \nabla G(X_{t_k}) \cdot \mathbb{E}\left[\sum_{t_k}^{t_{k+1}-1} (A_t - D_t^{\nabla G\max})\right] \rangle = \langle \nabla G(X_{t_k}) \cdot \mathbb{E}\left[\sum_{t_k}^{t_{k+1}-1} (A_t - D_{t_k}^{\nabla G\max} + D_{t_k}^{\nabla G\max} - D_t^{\nabla G\max})\right] \rangle$$
$$= \langle \nabla G(X_{t_k}) \cdot \mathbb{E}\left[\sum_{t_k}^{t_{k+1}-1} (A_t - D_{t_k}^{\nabla G\max})\right] \rangle + \langle \nabla G(X_{t_k}) \cdot E\left[\sum_{t_k}^{t_{k+1}-1} (D_{t_k}^{\nabla G\max} - D_t^{\nabla G\max})\right] \rangle \tag{47}$$

with:

$$\langle \nabla G(X_{t_k}) \cdot \mathbb{E}\left[\sum_{t_k}^{t_{k+1}-1} (A_t - D_{t_k}^{\nabla G\max})\right] \rangle = \langle \nabla G(X_{t_k}) \cdot \mathbb{E}[z_k](\Lambda - D_{t_k}^{\nabla G\max}) \rangle$$
$$\leq -\epsilon\, \mathbb{E}[z_k] \|\nabla G(X_{t_k})\| \tag{48}$$

where the (first) equation follows from classical reward-renewal arguments i.e. while the following inequality is obtained exploiting the same arguments as in proof of Theorem 5; in particular observe that $\langle \nabla G(X_t) \cdot \Lambda - D_t^{\nabla G\max}(I-R) \rangle = \langle \nabla G(X_t) \cdot \Lambda \rangle - \langle \nabla G(X_t) \cdot D_t^{\nabla G\max} \rangle$, and since by assumption Λ lies in the interior of $\mathcal{D}$, an $\epsilon' > 0$ can be found, such that also $\Lambda + \epsilon' D_t^{\nabla G\max}$ lies in $\mathcal{D}$; from which, recalling Lemma 3 we have: $\langle \nabla G(X_t) \cdot D_t^{\nabla G\max} \rangle = \max_{\substack{D \in \mathcal{D} \\ D \leq X_t}} \langle \nabla G(X_t) \cdot D \rangle = \max_{D \in \mathcal{D}} \langle \nabla G(X_n) \cdot D \rangle + o(\|\nabla G(X_t)\|) \geq \langle \nabla G(X_n) \cdot \Lambda + \epsilon' D_t^{\nabla G\max} \rangle \geq \langle \nabla G(X_n) \cdot \Lambda \rangle + \epsilon \|\nabla G(X_t)\|$, where the last inequality follows from (21).

The term:

$$\langle \nabla G(X_{t_k}) \cdot \mathbb{E}[\sum_{t=t_k}^{t_{k+1}-1} (D_{t_k}^{\nabla G\max} - D_t^{\nabla G\max})] \rangle = \mathbb{E}\left[\sum_{t=t_k}^{t_{k+1}-1} \langle \nabla G(X_{t_k}) \cdot D_{t_k}^{\nabla G\max} - D_t^{\nabla G\max} \rangle\right]$$
$$= \mathbb{E}\left[\sum_{t=t_k}^{t_{k+1}-1} \langle \nabla \big(G(X_{t_k}) - G(X_t) + G(X_t)\big) \cdot D_{t_k}^{\nabla G\max} - D_t^{\nabla G\max} \rangle\right]$$
$$= \mathbb{E}\left[\sum_{t=t_k}^{t_{k+1}-1} \langle \nabla \big(G(X_{t_k}) - G(X_t)\big) \cdot D_{t_k}^{\nabla G\max} - D_t^{\nabla G\max} \rangle\right] + \mathbb{E}\left[\sum_{t=t_k}^{t_{k+1}-1} \langle \nabla G(X_t) \cdot D_{t_k}^{\nabla G\max} - D_t^{\nabla G\max} \rangle\right] \tag{49}$$

$\langle \nabla \left(G(X_{t_k}) - G(X_t)\right) \cdot D_{t_k}^{\nabla G\max} - D_t^{\nabla G\max} \rangle = o(\|\nabla G(X_{t_k})\|)$ as an immediate consequence of (34); in this regard, we recall that by hypothesis polynomial moments $\mathbb{E}[||X_{t_k} - X_t||^h]$ are finite for any h; this again because $\{t_k\}$ is non defective sequence of stopping times and arrival vector is bounded. At last observe that the term $\langle \nabla G(X_t) \cdot D_{t_k}^{\nabla G\max} - D_t^{\nabla G\max} \rangle$ can not be positive, by construction.

As a conclusion

$$\mathbb{E}[G(X_{t_{k+1}}) \mid X_{t_k}] - G(X_{t_k}) \leq -\epsilon\, \mathbb{E}[z_k]\, \|\nabla G(X_{t_k})\| + o(\|\nabla G(X_{t_k})|)$$

Therefore (14). is satisfied, since for any $\epsilon'' < \epsilon\, \mathbb{E}[z_k]$, a $b > 0$ can be found such that

$$E\left[G(X_{t_{k+1}}) - G(X_{t_k}) \mid X_t\right] \leq -\epsilon'' \|\nabla G(X_{t_k})\|$$

for $\|X_t\| > b$.

At last, to show that (13) is satisfied too, observe that for any $Y_{t_k} : \|X_{t_k}\| \leq b$:

$$\mathbb{E}\left[G(X_{t_{k+1}})\right] = \mathbb{E}[G\big(X_{t_k} + \sum_{t_k}^{t_{k+1}-1} (A_t - D_t^{\nabla G\max})\big)\Big]$$
$$\overset{(46)}{=} G(X_{t_k}) + \langle \nabla G(X_{t_k}) \cdot \mathbb{E}\left[\sum_{t_k}^{t_{k+1}-1} A_t - D_t^{\nabla G\max}\right] \rangle + \sum_{i=2}^{h_0-1} \frac{1}{i!} \mathbb{E}\left[\left(\sum_{t_k}^{t_{k+1}-1} (A_t - D_t^{\nabla G\max})\right)^i\right] (\partial^i G)(X_k)$$
$$+ \frac{1}{h_0!} \mathbb{E}\left[(\sum_{t_k}^{t_{k+1}-1} (A_t - D_t^{\nabla G\max}))^{h_0} \cdot \left((\partial^{h_0} G)(X_{t_k} + \alpha \sum_{t_k}^{t_{k+1}-1} (A_t - D_t^{\nabla G\max})\right)\right]$$

can be easily shown to be bounded by $G(X_{t_k}) + v_0$ for an appropriate $v_0 > 0$, since i) $\mathbb{E}\left[\left(\sum_{t_k}^{t_{k+1}-1} (A_t - D_t^{\nabla G\max})\right)^i\right]$ are bounded for every i, as before; ii) $G(X_{t_k})$

and its derivatives $(\partial^i G)(X_k)$ are by assumption bounded over compact domains (in particular they are bounded over the domain $X : ||X|| \leq b$) because $G(X) \in C^\infty[\mathbb{R}^M \to \mathbb{R}]$; iii) $(\partial^{h_0} G)(X_{t_k} + \alpha \sum_{t_k}^{t_{k+1}-1}(A_t - D_t^{\nabla G\max}))$ is bounded as before in light of (22). The $\nabla G(X)$-stability of the system of queues immediately follows from Corollary 1 since $\lim_{\|X\|\to\infty} \nabla G(X) = \infty$ (as result of Lemma 1).

Proof of Corollary 2.

Consider the Lyapunov function $\mathcal{L}(X) = \frac{1}{h+1} G(X)^{h+1}$; denoting by $Z_t = A_t - D_t(I - R))$:

$$G(X_{t+1}) = G(X_t + Z_t) = G(X_t) + \langle G(X_t) \cdot Z_t \rangle + R^2(X_t, Z_t)$$

Now recalling (32) and (36), since by construction Z_t is a bounded norm vector we can claim that: $\langle G(X_t) \cdot Z_t \rangle = o(G(X_t))$, and $R^2(X_t, Z_t) = o(\|\nabla G(X_t))\|)$ as $X_t \to \infty$ Thus:

$$\mathbb{E}\left[\mathcal{L}(X_{t+1}) \mid X_t\right] \frac{1}{h+1} \mathbb{E}\left[\left(G(X_t + Z_t)\right)^{h+1}\right]$$
$$= \frac{1}{h+1} \mathbb{E}\left[\left(G(X_t) + \langle G(X_t) \cdot Z_t \rangle + o((\|\nabla G(X)\|\right)^{h+1}\right]$$
$$= \frac{1}{h+1} \left[\left(G(X_t)\right)^{h+1} + (h+1)\, \mathbb{E}[\langle G(X_t) \cdot Z_t \rangle] \left(G(X_t)\right)^h + o(\|\nabla G(X)\| \left(G(X_t)\right)^h\right]$$

Now considering X_t sufficiently large, such that $G(X_t)$ is positive (we recall that $G(X) \to \infty$, for $\|X\| \to \infty$ and thus, it must be positive outside some compact set), and taking the average, from (44) we have:

$$(G(X_t))^h \langle \nabla G(X_t) \cdot \Lambda_t - D_t^{\nabla G\max}(I - R) \rangle \quad \leq -\epsilon (G(X_t))^h \|\nabla G(X_t)\|$$

for some $\epsilon > 0$ (see (44)).

The $\|X\|^h$-stability immediately follows, observing that i) by construction $\lim_{\|X\|\to\infty} \frac{G^h(X)\|\nabla G(X)\|}{\|X^h\|} = \infty$; ii) for any $X_t : \|X_t\| \leq b$, $\frac{1}{h+1}\Big[G\big(X_t + Y_t)\big)^{h+1} - \big(G(X_t)\big)^{h+1}\Big]$ can be bounded by an appropriate constant v_0 (this because Y_t is bounded as well as $G(X)$ is bounded (from above and below) over compact sets);

The extension to the more general case can be carried out by observing that $\mathcal{L}(X) = \frac{1}{h+1} G(X)$ is a strong potential provided that of $G(X)$ is a strong potential by Corollary 3. Indeed denoting with $Z_{t_k} = \sum_{t_k}^{t_{k+1}-1} \big(A_t - D_t(I - R)\big)$, we have:

$$\mathbb{E}\left[\mathcal{L}(X_{t_{k+1}}) \mid X_t\right] = \mathcal{L}(X_{t_k}) + \mathbb{E}[\langle \nabla \mathcal{L}(X_{t_k}) \cdot Z_{t_k} \rangle] + \mathbb{E}[R^2_{\mathcal{L}}(X_{t_k}, Z_{t_k})]$$

with $\mathbb{E}[R^2_{\mathcal{L}}(X_{t_k}, Z_{t_k})] = o(\mathcal{L}(X_{t_k})))$ and $\mathbb{E}[\langle \mathcal{L}(X_{t_k}) \cdot Z_{t_k} \rangle] = o(\mathcal{L}(X_{t_k}))$ by (33) and (37), since, by construction, all polynomial moments of Y_{t_k} are finite.

Now observing that $\nabla \mathcal{L}(X_{t_k}) = \big(G(X)\big)^h \nabla G(X)$ we get:

$$\mathbb{E}\left[\mathcal{L}(X_{t_{k+1}}) \mid X_t\right] = \mathcal{L}(X_{t_k}) + \mathbb{E}[\big(G(X_{t_k})\big)^h \langle \nabla G(X_{t_k}) \cdot Z_{t_k} \rangle] + o(\| \big(G(X_{t_k})\big)^h \| \nabla G(X_{t_k}) \| \big)$$

The assertion follows along the same lines as before.

Proof of Corollary 4.

The proof is rather straightforward. Under i.i.d. arrivals and static constraints (i.e. when $\mathcal{H} = \mathcal{X}$) we can select the weak potential function $G(X)$, as a Lyapunov function, then:

From (30) and (36) we have:

$$\begin{aligned}\mathbb{E}&\left[G(X_t + A_t - D_t^{\nabla G \max}(I-R) \mid X_t\right] - G(X_t) \\ &\overset{(43)}{=} \langle \nabla G(X_t) \cdot \Lambda - D_t^{\nabla G \max}(I-R) \rangle + o(\| \nabla G(X_t) \|)\end{aligned}$$

with $\langle \nabla G(X_t) \cdot \Lambda - D_t^{\nabla G \max}(I-R) \rangle \overset{(44)}{\leq} -\epsilon' \| \nabla G(X_t) \|$ for an appropriate $\epsilon' > 0$.

Now again from (43), substituting $D^{\nabla G \mathrm{imp}}$ to $D^{\nabla G \max}$ we have:

$$\mathbb{E}[G(X_t + A_t - D^{\nabla G \mathrm{imp}}(I-R) \big| X_t] - G(X_t) = \langle \nabla G(X_t) \cdot \Lambda - D_t^{\nabla G \mathrm{imp}}(I-R) \rangle + o(\| \nabla G(X_t) \|)$$

and by assumption:

$$\langle \nabla G(X_t) \cdot D_t^{\nabla G \mathrm{imp}}(I-R) \rangle = \langle \nabla G(X_t) \cdot D_t^{\nabla G \max}(I-R) \rangle + o(\| \nabla G(X_t) \|)$$

Combining the two, we have:

$$\begin{aligned}\mathbb{E}\left[G\left(X_t + A_t - D^{\nabla G \mathrm{imp}}(I-R)\right) \mid X_t\right] - G(X_t) &= \langle \nabla G(X_t) \cdot \Lambda - D_t^{\nabla G \mathrm{imp}}(I-R) \rangle + o(\| \nabla G(X_t) \|) \\ &= \langle \nabla G(X_t) \cdot \Lambda - D_t^{\nabla G \max}(I-R) \rangle + o(\| \nabla G(X_t) \|)\end{aligned}$$

Thus

$$\mathbb{E}\left[G\left(X_t + A_t - D^{\nabla G \mathrm{imp}}(I-R)\right) \big| X_t\right] - G(X_t) \leq -\epsilon'' \| \nabla G(X_t)$$

for some $\epsilon'' > 0$. $|\nabla G(X)|$-stability for the system of queues follows. The proof in the case in which $G(X)$ is a strong potentials follows exactly along the same

lines. Furthermore adopting $\mathcal{L}(X) = \frac{1}{h+1}[G(X)]^{h+1}$ as a Lyapunov function and acting as before, the stability criteria can be strengthened.

Proof of Theorem 7.

Proof: First, we recall that in this case the space state of the DTMC is $Y_t = [X_t, D_t^{\nabla G\text{mem}}]$. We select the Lyapunov function

$$\mathcal{L}(Y_t) = \mathcal{L}(X_t, D_t^{\nabla G\text{mem}}) = \mathcal{L}_1(X_t) + \mathcal{L}_2(X_t, D_t^{\nabla G\text{mem}}),$$

with

$$\mathcal{L}_1(X_t) = G(X_t)$$

and

$$\mathcal{L}_2(X_t, D_t^{\nabla G\text{mem}}) = \left(\langle \nabla G(X_t) \cdot (D_t^{\nabla G\text{max}} - D_t^{\nabla G\text{mem}})(I - R)\rangle\right)^{\beta}$$

where $\beta > 1$ is given by (26); and we show that the drift condition of Theorem 4 is satisfied.

First, observe that $\mathcal{L}_2(X_t, D_t^{\nabla G\text{mem}})$ is well defined because $\langle \nabla G(X_t) \cdot (D_t^{\nabla G\text{max}} - D_t^{\nabla G\text{mem}})(I - R)\rangle \geq 0$ by construction.

Taking the second order Taylor expansion centered in X_t for $\mathcal{L}_1(X_{t+1})$ we get:

$$\mathcal{L}_1(X_{t+1}) = G(X_{t+1}) = G(X_t) + \langle \nabla G(X_t) \cdot A_t - D_t^{\nabla G\text{mem}}(I - R) \mid Y_t\rangle + R_G^2(X, A_t - D_t^{\nabla G\text{mem}}(I - R))$$

and recalling that $R_G^2(X, Y) = Y H_h(X_t + \alpha Y) Y^T = O(\|H_G(X_t)\|)$ for any vector Y in light of (19), we obtain:

$$\mathbb{E}[\mathcal{L}_1(X_{t+1}) - \mathcal{L}_1(X_t) \mid Y_t] = \mathbb{E}\left[\langle \nabla G(X_t) \cdot A_t - D_t^{\nabla G\text{mem}}(I - R) \mid Y_t\rangle\right] + O(\|H_G(X_t)\|)$$

Now

$$\begin{aligned}
\mathbb{E}\left[\langle \nabla G(X_t) \cdot A_t - D_t^{\nabla G\text{mem}}(I - R) \mid Y_t\rangle\right] &= \langle \nabla G(X_t) \cdot \Lambda - D_t^{\nabla G\text{mem}}(I - R)\rangle \\
&= \langle \nabla G(X_t) \cdot \Lambda - (D_t^{\nabla G\text{max}} - D_t^{\nabla G\text{max}} - D_t^{\nabla G\text{mem}})(I - R)\rangle \\
= \langle \nabla G(X_t) \cdot (D_t^{\nabla G\text{max}} - D_t^{\nabla G\text{mem}})(I - R)\rangle &+ \langle \nabla G(X_t) \cdot \Lambda - D_t^{\nabla G\text{max}}(I - R)\rangle \\
&\overset{(44)}{\leq} (\mathcal{L}_2(X_t, D_t^{\nabla G\text{mem}}))^{1/\beta} - \epsilon(\|\nabla G(X_t)\|)
\end{aligned}$$

for an appropriate $\epsilon > 0$. Indeed by assumption $\Lambda(I - R)^{-1}$ lies in the interior of $\mathcal{D}$.

Thus:

$$
\begin{aligned}
&\mathbb{E}[\mathcal{L}_1(X_{t+1}, D_{t+1}^{\nabla G\text{mem}}) - \mathcal{L}_1(X_t, D_t^{\nabla G\text{mem}}) \mid Y_t] \\
&\qquad \leq (\mathcal{L}_2(X_t, D_t^{\nabla G\text{mem}}))^{1/\beta} - \epsilon(\|\nabla G(X_t)\|) + O(\|H_G(X_t)\|) \qquad (50)
\end{aligned}
$$

Focusing instead on $\mathcal{L}_2(X_t, D_t^{\nabla G\text{mem}})$, we suppose for the moment $D_t \neq D_t^{\nabla G\text{max}}$:

$$
\begin{aligned}
&E\Big[\mathcal{L}_2(X_{t+1}, D_{t+1}^{\nabla G\text{mem}})\Big| X_t, D_t^{\nabla G\text{mem}} \neq D_t^{\nabla G\text{max}}\Big] \\
&= E\Big[\mathcal{L}_2(X_{t+1}, D_{t+1}^{\nabla G\text{mem}})\Big| X_t, D_{t+1}^{\nabla G\text{mem}} \neq D_{t+1}^{\nabla G\text{max}}, D_t^{\nabla G\text{mem}} \neq D_t^{\nabla G\text{max}}\Big] \cdot \\
&\qquad \Pr\Big\{D_{t+1}^{\nabla G\text{mem}} \neq D_{t+1}^{\nabla G\text{max}} \Big| X_t, D_t^{\nabla G\text{mem}} \neq D_t^{\nabla G\text{max}}\Big\} \\
&+ E\Big[\mathcal{L}_2(X_{t+1}, D_{t+1}^{\nabla G\text{mem}})\Big| X_t, D_{t+1}^{\nabla G\text{mem}} = D_{t+1}^{\nabla G\text{max}}, D_t^{\nabla G\text{mem}} \neq D_t^{\nabla G\text{max}}\Big] \cdot \\
&\qquad \Pr\Big\{D_{t+1}^{\nabla G\text{mem}} = D_{t+1}^{\nabla G\text{max}} \mid X_t, D_t^{\nabla G\text{mem}} \neq D_t^{\nabla G\text{max}}\Big\} \\
&\qquad \leq E\Big[\Big(\langle \nabla G(X_{t+1}) \cdot (D_{t+1}^{\nabla G\text{max}} - D_{t+1}^{\nabla G\text{mem}})(I-R)\rangle\Big)^\beta \Big| X_t, \\
&\qquad\qquad D_t^{\nabla G\text{mem}} \neq D_t^{\nabla G\text{max}}, D_{t+1}^{\nabla G\text{mem}} \neq D_{t+1}^{\nabla G\text{max}}\Big](1-\delta)
\end{aligned}
$$

where the last inequality comes from the fact that by construction:

$$
\begin{aligned}
&\mathbb{E}\left[\mathcal{L}_2(X_{t+1}, D_{t+1}^{\nabla G\text{mem}}) \mid X_t, D_{t+1}^{\nabla G\text{mem}} = D_{t+1}^{\nabla G\text{max}}, D_t^{\nabla G\text{mem}} \neq D_{t+1}^{\nabla G\text{max}}\right] \\
&\qquad = \mathbb{E}\left[\left(\langle \nabla G(X_{t+1}) \cdot (D_{t+1}^{\nabla G\text{max}} - D_{t+1}^{\nabla G\text{max}})(I-R)\rangle\right)^\beta\right] = 0
\end{aligned}
$$

while $\Pr\{D_{t+1}^{\nabla G\text{mem}} \neq D_t^{\nabla G\text{max}} \mid X_t, D_t^{\nabla G\text{mem}} \neq D_t^{\nabla G\text{max}}\} \leq 1-\delta$.

Now since our scheme guarantees that: $\langle \nabla G(X_{t+1}) \cdot D_{t+1}^{\nabla G\text{mem}}(I-R)\rangle \geq \langle \nabla G(X_{t+1}) \cdot D_t^{\nabla G\text{mem}}(I-R)\rangle$ we can write:

$$
\begin{aligned}
&\mathbb{E}\left[\left(\langle\nabla G(X_{t+1})\cdot(D_{t+1}^{\nabla G\max}-D_{t+1}^{\nabla G\mathrm{mem}})(I-R)\rangle\right)^{\beta}\Big|X_t, D_t^{\nabla G\mathrm{mem}}\neq D_t^{\nabla G\max}, D_{t+1}^{\nabla G\mathrm{mem}}\neq D_{t+1}^{\nabla G\max}\right](1-\delta)\\
&\leq \mathbb{E}\left[\left(\langle\nabla G(X_{t+1})\cdot(D_{t+1}^{\nabla G\max}-D_{t}^{\nabla G\mathrm{mem}})(I-R)\rangle\right)^{\beta}\Big|X_t, D_t^{\nabla G\mathrm{mem}}\neq D_t^{\nabla G\max}\right](1-\delta)\\
&= \mathbb{E}\left[\left(\langle\nabla G(X_{t+1})\cdot\big(D_{t+1}^{\nabla G\max}-D_{t}^{\nabla G\max}+D_{t}^{\nabla G\max}-D_{t}^{\nabla G\mathrm{mem}}\big)(I-R)\rangle\right)^{\beta}\Big|X_t,\right.\\
&\qquad\qquad \left. D_t^{\nabla G\mathrm{mem}}\neq D_t^{\nabla G\max}\right](1-\delta)\\
&\qquad = \mathbb{E}\left[\left(\langle\nabla G(X_{t+1})\cdot\big(D_{t+1}^{\nabla G\max}-D_{t}^{\nabla G\max}\big)(I-R)\rangle\right.\right.\\
&\qquad\quad \left.\left. +\langle\nabla G(X_{t+1})\cdot\big(D_{t}^{\nabla G\max}-D_{t}^{\nabla G\mathrm{mem}}\big)(I-R)\rangle\right)^{\beta}\mid X_t,\right.\\
&\qquad\qquad\qquad\qquad \left. D_t\neq D_t^{\nabla G\max}\right](1-\delta)
\end{aligned}
$$

Expanding component-wise in Taylor series $\nabla G(X)$, we can write, similarly as before, $\nabla G(X_{t+1}) = \nabla G(X_t + A_t - D_t^{\nabla G\mathrm{mem}}(I-R)) = \nabla G(X_t) + O(\|H_G(X_t)\|)$ in light of (19), (34) and of the fact that both A_t and $D_t^{\nabla G\mathrm{mem}}$ are bounded. Furthermore observe that by construction: $\langle\nabla G(X_t)\cdot(D_t^{\nabla G\max} - D_{t+1}^{\nabla G\max})(I-R)\rangle \geq 0$, therefore we obtain:

$$
\begin{aligned}
&\mathbb{E}[\mathcal{L}_2(X_{t+1}, D_{t+1}^{\nabla G\mathrm{mem}}) \mid Y_t \text{ with } D_t^{\nabla G\mathrm{mem}}\neq D_t^{\nabla G\max}]\\
&\leq \left(\langle\nabla G(X_t)\cdot\big(D_t^{\nabla G\max}-D_t^{\nabla G\mathrm{mem}}\big)(I-R)\rangle(1-\delta)+O(\|H_G(X_t)\|)\right)^{\beta}\\
&\qquad = \mathcal{L}_2(X_t, D_t^{\nabla G\mathrm{mem}})(1-\delta)+o(\mathcal{L}_2(X_t, D_t^{\nabla G\mathrm{mem}}))
\end{aligned}
$$

Thus:

$$
\begin{aligned}
&\mathbb{E}[\mathcal{L}_2(X_{t+1}, D_{t+1}^{\nabla G\mathrm{mem}}) \mid X_t, D_t^{\nabla G\mathrm{mem}}\neq D_t^{\nabla G\max}] - \mathcal{L}_2(X_t, D_t^{\nabla G\mathrm{mem}})\\
&\qquad \leq -\delta\mathcal{L}_2(X_t, D_t^{\nabla G\mathrm{mem}}) + o(\mathcal{L}_2(X_t, D_t^{\nabla G\mathrm{mem}})) \qquad (51)
\end{aligned}
$$

while for $D_t^{\nabla G\mathrm{mem}} = D_t^{\nabla G\max}$:

$$
\begin{aligned}
&\mathbb{E}[\mathcal{L}_2(X_{t+1}, D_{t+1}^{\nabla G\mathrm{mem}}) \mid X_t, D_t^{\nabla G\mathrm{mem}} = D_t^{\nabla G\max}]\\
&= \mathbb{E}\left[\left(\langle\nabla G(X_{t+1})\cdot(D_{t+1}^{\nabla G\max}-D_t^{\nabla G\max})(I-R)\rangle\right)^{\beta}\right] = \mathbb{E}\left[\left(\langle\nabla G(X_t)\cdot(D_{t+1}^{\nabla G\max}-D_t^{\nabla G\max})(I-R)\rangle\right.\right.\\
&\qquad \left.\left. + O\Big(\|H_G(X_t)\|\Big)\right)^{\beta}\right] \leq O(\|H_G(X_t)\|^{\beta})) \qquad (52)
\end{aligned}
$$

In light of the fact that $\langle\nabla G(X_t)\cdot(D_{t+1}^{\nabla G\max} - D_t^{\nabla G\max})(I-R)\rangle$ is negative by construction, and $(x)^\beta$ is monotone; indeed we recall that, by construction: $\langle\nabla G(X_{t+1})\cdot(D_{t+1}^{\nabla G\max} - D_t^{\nabla G\max})(I-R) \geq 0$.

Combining together (50) and (51) or (52), (we recall that $\mathcal{L}_2(X_t, D_t^{\nabla G\mathrm{mem}}) = 0$ if$D_t^{\nabla G\mathrm{mem}} = D_t^{\nabla G\mathrm{max}}$) we obtain:

$$\begin{aligned} &\mathbb{E}[\mathcal{L}(X_{t+1}, D_{t+1}^{\nabla G\mathrm{mem}}) \mid Y_t] \\ &\le -\epsilon(\|\nabla G(X_t)\|)+(\mathcal{L}_2(X_t, D_t^{\nabla G\mathrm{mem}}))^{1/\beta}-\delta\mathcal{L}_2(X_t, D_t^{\nabla G\mathrm{mem}})+o(\mathcal{L}_2(X_t, D_t^{\nabla G\mathrm{mem}}))+O(\|H_G(X_t)\|^{\beta}) \\ &= -\epsilon(\|\nabla G(X_t)\|) - \delta\mathcal{L}_2(X_t, D_t^{\nabla G\mathrm{mem}}))+ \end{aligned} \quad (53)$$

in light of (26). Thus for a sufficiently large $b > 0$, we can claim that:

$$\mathbb{E}[\mathcal{L}(X_{t+1}, D_{t+1}^{\nabla G\mathrm{mem}}) \mid Y_t] - \mathcal{L}(X_t, D_t^{\nabla G\mathrm{mem}}) \le -\epsilon'(\|\nabla G(X_t)\|)$$

for any $Y_t]$, such that $\|X_t\| > b$ and for any $\epsilon' < \epsilon$ (in this regard, we recall that β is selected in such a way to guarantee that $\|H_G(X_t)\|^{\beta} = o(\|\nabla G(X_t)\|)$).

$|\nabla G(X)|$-stability for the system of queues follows, since for any $Y_t : \|X_t\| \le b$, $\mathcal{L}(Y_{t+1}) - \mathcal{L}(Y_t)$ is bounded, as immediate consequence of the fact that $\|X_{t+1} - X_t\|$ is bounded.

The stability criteria can be strengthened. For any $h \in \mathbb{N}$, we can prove that the system of queues is $\|X^h\|$-stable under any admissible arrival vector, by selecting the Lyapunov function $\mathcal{L}'(Y_t) = \mathcal{L}'(X_t, D_t^{\nabla G\mathrm{mem}}) = \frac{1}{h+1}\big(\mathcal{L}(X_t, D_t^{\nabla G\mathrm{mem}})\big)^{h+1} = \frac{1}{h+1}\big(\mathcal{L}_1(X_t) + \mathcal{L}_2(X_t, D_t^{\nabla G\mathrm{mem}})\big)^{h+1} = \frac{1}{h+1}\Big(G(X_t) + \big(\langle \nabla G(X_t) \cdot (D_t^{\nabla G\mathrm{max}} - D_t^{\nabla G\mathrm{mem}})(I-R)\rangle\big)^{\beta}\Big)^{h+1}$. The derivation proceeds along the same lines as the proof of Corollary 2, essentially showing that $E\big[\mathcal{L}'(Y_{t+1}) \mid Y_t\big] - \mathcal{L}'(Y_t) \approx \big(\mathcal{L}(X_t, D_t^{\nabla G\mathrm{mem}})\big)^h \big(E\big[\mathcal{L}(X_{t+1}, D_{t+1}^{\nabla G\mathrm{mem}}) \mid Y_t\big] - \mathcal{L}(X_t, D_t^{\nabla G\mathrm{mem}})\big) \le -\epsilon(\|\nabla G(X_t)\|)\big(\mathcal{L}(X_t, D_t^{\nabla G\mathrm{mem}})\big)^h$, for any Y_t : $\|X_t\| > b$ with a sufficiently large $b > 0$, and for a sufficiently small $\epsilon > 0$.

Proof of Theorem 8.

This proof combines arguments already applied in the proofs of Theorem 6 and Theorem 7. First, we observe that the state of the DTMC representing the system evolution is given by vector: $Y_t = [X_t, S_t, S_t^M]$, where $S_t \in \mathcal{S}^A \times \mathcal{S}^D$ represents the dynamic of exogenous arrivals, and service constraints, while S_t^M provides the additional information about the memory state of the scheduling algorithm; such information correspond to the set of departure vectors $D_t^M(S^D)$, memorized by the scheduling algorithm. We recall that, by construction, $D_t^M(S^D)$ is the departure vector employed at the last occurrence of state S^D before t.

We select a Lyapunov function with a similar structure as the one used in Theorem 7, however this time, things are made slightly more difficult by the fact that the memory of the scheme is significantly larger. Furthermore the stability properties

of Y_t are derived from those of the DTMC Y_{t_k} obtained through the sub-sampling of Y_t, at instants $\{t_k\}$ in which the DTMC $S_{t_k} = S_0$, for a particular state S_0 (Corollary 1). Again we can claim that $\{t_k\}$ forms a sequence of non defective regeneration points for the system since S_t is finite state and ergodic.

In more detail the selected Lyapunov function is:

$$\mathcal{L}(Y_t) = \mathcal{L}(X_t, S_t^M) = \mathcal{L}_1(X_t) + \mathcal{L}_2(X_t, S_t^M),$$

with:

$$\mathcal{L}_1(X_t) = G(X_t)$$

and

$$\mathcal{L}_2(X_t, S_t^M) = \sum_{S \in \mathcal{S}} \pi_S \left(\langle \nabla G(X_t) \cdot (D_t^{\nabla G \max}(S) - D_t^M(S))(I - R) \rangle \right)^{\beta},$$

where, once again, we recall that $D_t^M(S)$ represents the memorized departure vector that corresponds to state S (i.e., to the component S^D of S associated with service constraints), $\beta > 1$ is specified by (26) and π_S is the steady state probability of the DTMC S_t governing arrivals and dynamic constraint conditions.

In the remainder of the proof to simplify the notation we omit the dependency of the departure vector on constraints conditions, writing $D_t^{\nabla G \mathrm{mem}}$ instead of $D_t^{\nabla G \mathrm{mem}}(S_t^D)$, whenever this can be done without causing confusion.

Taking the second order Taylor expansion centered in X_{t_k} for $G(X_{t_{k+1}})$ we get:

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{L}_1(X_{t_{k+1}}) - \mathcal{L}_1(X_{t_k}) \mid Y_t\right] \\
&\quad = \mathbb{E}\left[\sum_{t=t_k}^{t_{k+1}-1} \langle \nabla G(X_t) \cdot A_t - D_t^{\nabla G \mathrm{mem}}(I - R) \rangle \mid Y_{t_k}\right] + O(\|H_G(X_t)\|)
\end{aligned}$$

with:

$$\begin{aligned}
&\mathbb{E}\left[\langle \nabla G(X_{t_k}) \cdot \sum_{t=t_k}^{t_{k+1}-1} A_t - D_t^{\nabla G \mathrm{mem}}(I - R) >\mid Y_{t_k}\right] \\
&= \mathbb{E}\left[\langle \nabla G(X_{t_k}) \cdot \sum_{t=t_k}^{t_{k+1}-1} A_t - (D_t^{\nabla G \max} - D_t^{\nabla G \max} + D_t^{\nabla G \mathrm{mem}})(I - R) \rangle \mid Y_{t_k}\right] \\
&\qquad = \mathbb{E}\left[\langle \nabla G(X_{t_k}) \cdot \sum_{t=t_k}^{t_{k+1}-1} (D_t^{\nabla G \max} - D_t^{\nabla G \mathrm{mem}})(I - R) \rangle \mid Y_{t_k}\right] \\
&\qquad\qquad + \mathbb{E}\left[\langle \nabla G(X_{t_k}) \cdot \sum_{t=t_k}^{t_{k+1}-1} A_t - D_t^{\nabla G \max}(I - R) \rangle \mid Y_{t_k}\right]
\end{aligned}$$

Now, $\mathbb{E}\left[\sum_{t=t_k}^{t_{k+1}-1}\langle\nabla G(X_t) \cdot A_t - D_t^{\nabla G\max}(I - R)\rangle \mid Y_{t_k}\right] \leq -\epsilon\,\mathbb{E}[z_k](\|\nabla G(X_t)\|)$, from (47) and (48) in the proof of Theorem 6.

While:

$$\mathbb{E}\left[\langle\nabla G(X_{t_k}) \cdot \sum_{t=t_k}^{t_{k+1}-1}(D_t^{\nabla G\max} - D_t^{\nabla G\mathrm{mem}})(I-R)\rangle \mid Y_{t_k}\right]$$

$$\mathbb{E}\left[\langle\nabla G(X_{t_k})\cdot \sum_{t=t_k}^{t_{k+1}-1}(D_t^{\nabla G\max} - D_t^{\nabla G\mathrm{mem}} + D_{t_k}^{\nabla G\max}(S_t^D) - D_{t_k}^M(S_t^D) - D_{t_k}^{\nabla G\max}(S_t^D) + D_{t_k}^M(S_t^D))(I-R)\rangle \mid Y$$

$$\leq \mathbb{E}\left[\langle\nabla G(X_{t_k})\cdot \sum_{t=t_k}^{t_{k+1}-1}(D_{t_k}^{\nabla G\max}(S_t^D) - D_{t_k}^M(S_t^D))(I-R)\rangle \mid Y_{t_k}\right] + O(\|H_G(X_t)\|)$$

where we recall that

$$D_{t_k}^{\nabla G\max}(S_t^D) = \underset{\substack{D\in\mathcal{D}(S_t^D)\\ D\leq X_{t_k}}}{\arg\max}\langle\nabla G(X_{t_k})\cdot D(I-R)\rangle$$

and $D_{t_k}^M(S_t^D)$ is the departure vector memorized by the scheduling at time t_k, which corresponds to state S_t^D. Observe, indeed, that $\mathbb{E}\left[\langle\nabla G(X_{t_k}) \cdot \sum_{t=t_k}^{t_{k+1}-1}(D_t^{\nabla G\max} - D_{t_k}^{\nabla G\max}(S_t^D))(I-R)\rangle \mid Y_{t_k}\right] \leq 0$ from the definition of $D_{t_k}^{\nabla G\max}(S_t^D)$ and $\mathbb{E}\left[\langle\nabla G(X_{t_k}) \cdot \sum_{t=t_k}^{t_{k+1}-1} D_{t_k}^M(S_t^D) - D_t^{\nabla G\mathrm{mem}})(I-R)\rangle \mid Y_{t_k}\right] = \mathbb{E}\left[\sum_{t=t_k}^{t_{k+1}-1}\langle\nabla G(X_t) \cdot D_{t_k}^M(S_t^D) - D_t^{\nabla G\mathrm{mem}})(I-R)\rangle \mid Y_{t_k}\right] + O(\|H_G(X_t)\|)$, with $\langle\nabla G(X_t)\cdot(D_{t_k}^M(S_t^D) - D_t^{\nabla G\mathrm{mem}})(I-R)\rangle \leq 0$ from (28).

Now:

$$\begin{aligned}
&\mathbb{E}\left[\langle\nabla G(X_{t_k})\cdot\sum_{t=t_k}^{t_{k+1}-1}(D_{t_k}^{\nabla G\max}(S_t) - D_{t_k}^M(S_t))(I-R)\rangle \mid Y_{t_k}\right]\\
&\qquad = \mathbb{E}[z_k]\sum_{S\in\mathcal{S}}\pi_S\langle\nabla G(X_{t_k})\cdot(D_{t_k}^{\nabla G\max}(S) - D_{t_k}^M(S))(I-R)\rangle\\
&\qquad\qquad = \mathbb{E}[z_k]\mathcal{L}_2(X_{t_k}, S_{t_k}^M))^{1/\beta}
\end{aligned}$$

Thus:

$$\mathbb{E}[\mathcal{L}_1(X_{t_{k+1}}) - \mathcal{L}_1(X_{t_k}) \mid Y_{t_k}] \leq \mathbb{E}[z_k](\mathcal{L}_2(X_{t_k}, S_{t_k}^M))^{1/\beta} - \epsilon\|(\nabla G(X_{t-k}))\| + O(\|H_G(X_t)\|) \tag{54}$$

Focusing instead on $\mathcal{L}_2(X_{t_k}, S_{t_k}^M)$, first observe that $\forall t \in [t_k, t_{k+1}]$, stopping time for the Markov Chain S_t^D, we have:

$$\begin{aligned}
&\mathbb{E}\left[\left(\langle \nabla G(X_{t_{k+1}}) \cdot (D_{t_{k+1}}^{\nabla G\max}(S) - D_{t_{k+1}}^M(S))(I-R)\rangle\right)^\beta \mid D_t^M(S) = D_t^{\nabla G\max}(S)\right] \\
&\qquad = \mathbb{E}\left[\left(\langle \nabla G(X_{t_{k+1}}) \cdot (D_{t_{k+1}}^{\nabla G\max}(S) - D_t^{\nabla G\max}(S))(I-R)\rangle\right.\right. \\
&+ \left.\left.\langle \nabla G(X_{t_{k+1}}) \cdot (D_t^{\nabla G\max}(S) - D_{t_{k+1}}^M(S))(I-R)\rangle\right)^\beta \mid D_t^M(S) = D_t^{\nabla G\max}(S)\right] \\
&= \mathbb{E}\left[\left(\langle \nabla G(X_{t_{k+1}}) \cdot (D_{t_{k+1}}^{\nabla G\max}(S) - D_t^{\nabla G\max}(S))(I-R)\rangle + \langle \nabla G(X_{t_{k+1}}) \cdot (D_t^M(S) - D_{t_{k+1}}^M(S))(I-R)\rangle\right)^\beta\right] \\
&\qquad \le \mathbb{E}\left[\left(\langle \nabla G(X_{t_{k+1}}) \cdot (D_{t_{k+1}}^{\nabla G\max}(S) - D_t^{\nabla G\max}(S))(I-R)\rangle\right)^\beta\right] \qquad (55)
\end{aligned}$$

because by construction $\langle \nabla G(X_{t_{k+1}}) \cdot (D_t^M(S) - D_{t_{k+1}}^M(S))(I-R) \le 0$ and $(x)^\beta$ is monotone (observe that by construction the arguments of the power operator is positive). Now:

$$\begin{aligned}
&\mathbb{E}\left[\left(\langle \nabla G(X_{t_{k+1}}) \cdot (D_{t_{k+1}}^{\nabla G\max}(S) - D_t^{\nabla G\max}(S))(I-R)\rangle\right)^\beta\right] \\
&\qquad = \mathbb{E}\left[\left(\langle \nabla G(X_t) \cdot (D_{t_{k+1}}^{\nabla G\max} - D_t^{\nabla G\max})(I-R)\rangle\right.\right. \\
&\quad \left.\left.+ O\Big(\|H_G(X_t)\|\Big)\right)^\beta\right] \le O\Big(\|H_G(X_t)\|\Big)^\beta = O\Big(\|H_G(X_{t_{k+1}})\|\Big)^\beta \qquad (56)
\end{aligned}$$

where first equation is a direct consequence of (35) and the fact that polynomial moments of $t_{k+1} - t$ are finite, the following inequality is a consequence of the fact that $\langle \nabla G(X_t) \cdot (D_{t_{k+1}}^{\nabla G\max}(S) - D_t^{\nabla G\max}(S))(I-R)\rangle$ is negative by construction, and $(x)^\beta$ is monotone Thus combining (55) and (56) $\forall t \in [t_k, t_{k+1}]$, stopping time for the Markov Chain S_t^D, we get:

$$\mathbb{E}\left[\left(\langle \nabla G(X_{t_{k+1}}) \cdot (D_{t_{k+1}}^{\nabla G\max}(S) - D_{t_{k+1}}^M(S))(I-R)\rangle\right)^\beta\right] \mid D_t^M(S) = D_t^{\nabla G\max}(S)\Big] = O\Big(\|H_G(X_t\|\Big)^\beta \qquad (57)$$

From which we can conclude that:

$$\mathbb{E}\left[\left(\langle\nabla G(X_{t_{k+1}})\cdot(D^{\nabla G\max}_{t_{k+1}}(S)-D^M_{t_{k+1}}(S))(I-R)\rangle\right)^\beta \mid D^M_{t_k}(S)\neq D^{\nabla G\max}_{t_k}(S)\right]=$$
$$\mathbb{E}\left[\left(\langle\nabla G(X_{t_{k+1}})\cdot(D^{\nabla G\max}_{t_{k+1}}(S)-D^M_{t_{k+1}}(S))(I-R)\rangle\right)^\beta \mid D^M_{t}(S)\neq D^{\nabla G\max}_{t}(S)\ \forall t\in[t_k,t_{k+1}]\right]\cdot$$
$$\Pr\left\{D^M_t(S)\neq D^{\nabla G\max}_t(S)\ \forall t\in(t_k,t_{k+1}]\mid D^M_{t_k}(S)\neq D^{\nabla G\max}_{t_k}(S)\right\}+$$
$$\mathbb{E}\left[\left(\langle\nabla G(X_{t_{k+1}})\cdot(D^{\nabla G\max}_{t_{k+1}}(S)-D^M_{t_{k+1}}(S))(I-R)\rangle\right)^\beta \mid \exists t\in(t_k,t_{k+1}]:\ D^M_t(S)=D^{\nabla G\max}_t(S),\ D^M_{t_k}(S)\neq D^{\nabla G\max}_{t_k}(S)\right]\cdot$$
$$\Pr\left\{\exists t\in(t_k,t_{k+1}]:\ D^M_t(S)=D^{\nabla G\max}_t(S)\mid D^M_{t_k}(S)\neq D^{\nabla G\max}_{t_k}(S)\right\}=$$
$$\mathbb{E}\left[\left(\langle\nabla G(X_{t_{k+1}})\cdot(D^{\nabla G\max}_{t_{k+1}}(S)-D^M_{t_{k+1}}(S))(I-R)\rangle\right)^\beta \mid D^M_{t}(S)\neq D^{\nabla G\max}_{t}(S)\ \forall t\in[t_k,t_{k+1}]\right](1-\delta\hat{\pi}_S)+O(\|H_G(X_{tk})\|) \quad (58)$$

where $\hat{\pi}_S$ denotes the probability state S is visited in $(t_k,t_{k+1}]$ (i.e., between to following visits to state S_0). Indeed observe that by construction $\Pr\left\{D^M_t(S)\neq D^{\nabla G\max}_t(S)\ \forall t\in(t_k,t_{k+1}]\mid D^M_{t_k}(S)\neq D^{\nabla G\max}_{t_k}(S)\right\}\leq 1-\delta\hat{\pi}_S$ since $D^M_t(S) = D^{\nabla G\max}_t$ with a probability greater than δ provided that state S has been visited at time $t=(t_k,t_{k+1}]$, and we can apply (57) to $\mathbb{E}\left[\left(\langle\nabla G(X_{t_{k+1}})\cdot(D^{\nabla G\max}_{t_{k+1}}(S)-D^M_{t_{k+1}}(S))(I-R)\rangle\right)^\beta \mid \exists t\in(t_k,t_{k+1}]:\ D^M_t(S)=D^{\nabla G\max}_t(S), D^M_{t:k}(S)\neq D^{\nabla G\max}_{t_k}(S)\right]$.

At last

$$\begin{aligned}\mathbb{E}\Big[\left(\langle\nabla G(X_{t_{k+1}})\cdot(D^{\nabla G\max}_{t_{k+1}}(S)-D^M_{t_{k+1}}(S))(I-R)\rangle\right)^\beta \\ \mid \forall t\in[t_k,t_{k+1}]:\ D^M_t(S)\neq D^{\nabla G\max}_t(S)\Big] \\ \leq \mathbb{E}\left[\left(\langle\nabla G(X_{t_{k+1}})\cdot(D^{\nabla G\max}_{t_{k+1}}(S)-D^M_{t_k}(S))(I-R)\rangle\right)^\beta\right. \\ \leq \mathbb{E}\left[\left(\langle\nabla G(X_{t_k})\cdot(D^{\nabla G\max}_{t_k}(S)-D^M_{t_k}(S))(I-R)\rangle+O\big(\|H_G(X_t)\|\big)\right)^\beta\right]\end{aligned} \quad (59)$$

where the first inequality derive from (28) while the following inequality can be obtained expanding in Taylor series $G(X_{t_{k+1}})$ around X_{t_k} and observing that by construction $\nabla G(X_{t_k})\cdot(D^{\nabla G\max}_{t_k}(S)-D^{\nabla G\max}_{t_{k+1}}(S))(I-R)\leq 0$.

Thus combining (58) and (59) we get:

$$\mathbb{E}\left[\left(\langle\nabla G(X_{t_{k+1}})\cdot(D^{\nabla G\max}_{t_{k+1}}(S)-D^{M}_{t_{k+1}}(S))(I-R)\rangle\right)^{\beta}\mid D^{M}_{t_k}(S)\neq D^{\nabla G\max}_{t_k}(S)\right]$$
$$\leq \mathbb{E}\left[\left(\langle\nabla G(X_{t_k})\cdot(D^{\nabla G\max}_{t_k}(S)-D^{M}_{t_k}(S))(I-R)(1-\delta\hat{\pi}_S)\rangle\right)^{\beta}\right]+o(\langle\nabla G(X_{t_k})\cdot(D^{\nabla G\max}_{t_k}(S)-D^{M}_{t_k}(S))(I-R$$

Now multiplying for π_S and summing over all the states and recalling (60) and (57) and recalling that $\nabla G(X_{t_k})\cdot(D^{\nabla G\max}_{t_k}(S)-D^{M}_{t_k}(S))(I-R)\rangle = 0$ if $D_{t_k}(S) = D^{\nabla G\max}_{t_k}(S)$, we get:

$$\begin{aligned}
&\mathbb{E}[\mathcal{L}_2(X_{t_{k+1}}, S^M_{t_{k+1}}) \mid X_{t_k}, S^M_{t_k}] \\
&= \sum_S \pi_S\, \mathbb{E}\left[\left(\langle\nabla G(X_{t_{k+1}})\cdot(D^{\nabla G\max}_{t_{k+1}}(S)-D^{M}_{t_{k+1}}(S))(I-R)\rangle\right)^{\beta}\mid X_{t_k}, S^M_{t_k}\right] \\
&\quad \leq \sum_S \pi_S(1-\delta\hat{\pi}_S)\left[\left(\langle\nabla G(X_{t_k})\cdot(D^{\nabla G\max}_{t_k}(S)-D^{M}_{t_k}(S))(I-R)\rangle\right)^{\beta}\right] \\
&+o\Big(\sum_S \pi_S\big(\langle\nabla G(X_{t_k})\cdot(D^{\nabla G\max}_{t_k}(S)-D^{M}_{t_k}(S))(I-R)\rangle\big)^{\beta}\Big)+o(\|H_G(X_{t_k})\|^{\beta}) \\
&\leq \mathbb{E}[\mathcal{L}_2(X_{t_k}, S^M_{t_k})](1-\delta\min_S \pi_S\hat{\pi}_S)) + o(\mathcal{L}_2(X_{t_k}, S^M_{t_k})) + o(\|H_G(X_{t_k})\|^{\beta})
\end{aligned} \tag{61}$$

where observe that $\min_S \pi_S\hat{\pi}_S > 0$ since S^D_t is a finite state ergodic Markov Chain.

At last, by combining (61) and (54), we can easily show that drift condition (14) is satisfied.

$|\nabla G(X)|$-stability for the system of queues easily follows, since (13) can be easily derived (as for the previous Theorems) from the following three facts: i) $\mathcal{L}(Y)$ and is indefinitely continuously derivable, and thus bounded (along with its derivatives) over compact domains; ii) at any instant t both arrival and departure vectors are bounded; iii) $\mathbb{E}[z^i_k]$ are bounded for any $i > 0$.

We further notice that the stability criteria can be strengthened. For any $h \in \mathbb{N}$, we can prove that the system of queues is $\|X^h\|$-stable under any admissible arrival vector, by selecting the Lyapunov function $\mathcal{L}'(Y_t) = \mathcal{L}'(X_t, D^{\nabla G\mathrm{mem}}_t) = \frac{1}{h+1}\big(\mathcal{L}(X_t, D^{\nabla G\mathrm{mem}}_t)\big)^{h+1} = \frac{1}{h+1}\big(\mathcal{L}_1(X_t) + \mathcal{L}_2(X_t, D^{\nabla G\mathrm{mem}}_t)\big)^{h+1}$.